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Backward Deletion to Minimize Prediction Errors in Models from Factorial Experiments with Zero to Six Center Points

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Prepared for the
One-hundred-fortieth Annual Meeting
of the American Statistical Association
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CONTENTS

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SUMMARY

Population model coefficients were chosen to simulate a saturated 2^4 fixed-effects experiment having an unfavorable distribution of relative values. Using random number studies, deletion strategies were compared that were based on the F-distribution, on an order statistics distribution of Cochran's, and on a combination of the two. The strategies were compared under the criterion of minimizing the maximum prediction error, wherever it occurred, among the two-level factorial points. The strategies were evaluated for each of the conditions of 0, 1, 2, 3, 4, 5, or 6 center points.

Three classes of strategies were identified as being appropriate, depending on the extent of the experimenter's prior knowledge. In almost every case the best strategy was found to be unique according to the number of center points. Among the three classes of strategies, a security regret class of strategy was demonstrated as being widely useful in that over a range of coefficients of variation from 4 to 65 percent, the maximum predictive error was never increased by more than 12 percent over what it would have been, if the best strategy had been used for the particular coefficient of variation.

The relative efficiency of the experiment, when using the security regret strategy, was examined as a function of the number of center points, and was found to be best when the design used one center point.

INTRODUCTION

The two-level, fixed effects, full or fractional-factorial designs of experiment, without replication, are often appropriate for those situations where the experiment is very expensive or time consuming. An example of costly experimenting is provided by the destructive testing of simulated aircraft turbine engine components, as in the rotor burst protection testing described by Mangano (1977). A rotor burst protection investigation was planned as a two-level fractional factorial experiment to measure the containment efficiencies of composite structures. The description (Holms 1977a) of that experiment is illustrative of one area of applicability of the results of the present investigation.

If a two-level full- or fractional-factorial experiment is performed and n_c observations are obtained from n_c orthogonal experimental conditions, the appropriate empirical equation for representing the results can have as many as n_c terms, each with a coefficient that has been fitted to the data. When this

is done, a question that should be asked is: "Can the predictive accuracy be improved if some of the terms are deleted?" The fact that some of the terms might degrade the predictive accuracy of a fitted equation was recognized by Walls and Weeks (1969) but they gave no procedure for identifying such terms.

A method for the sequential deletion of terms that was intended to reduce the prediction error was given by Kennedy and Bancroft (1971). Their method assumed that the experimenter has a prior established order for subjecting the terms to a sequence of significance tests. Unfortunately, in many experimental situations, there is no subject matter basis for establishing a prior order, and in such cases an order statistics procedure is appropriate. An order statistics approach for significance testing was used in a pair of related papers by Daniel (1959) and by Birnbaum (1959). They were not then seeking to minimize prediction errors.

For model selection procedures used with small saturated experiments (experiments designed to have only as many experimental conditions as there are model parameters to be fitted), the analysis should begin with a minimum number of estimable terms being sacrificed to form a denominator for the test statistic. A procedure using m-terms sacrificed, where m can be as small as one, was investigated by Holms and Berrettoni (1969). The object was to delete terms in a manner where some control was maintained over the probabilities of Type 1 or Type 2 decision errors.

The minimizing of prediction error was the object of an investigation of a chain pooling strategy as described by Holms (1974). Whereas that investigation had assumed that only one cycle of analysis would be used, a suggestion given by Holms and Berrettoni (1969) was that more than one cycle should be used. The purpose of a further chain pooling investigation (Holms (1977b)) was to optimize a combined procedure that might contain more than one analysis cycle, where the procedure is to be optimized for minimum prediction error. An important application of chain pooling occurs in empirical optimum seeking.

A widely accepted methodology for the design and analysis of experiments that are efficient for the empirical attainment of optimum conditions was introduced by Box and Wilson (1951) and refined by Box and Hunter (1957). These methods are now known as response surface methodology. Designs that are optimal for fitting second degree equations were studied by Lucas (1974 and 1976), who was concerned with the optimality of single block designs, but multi-block designs are often appropriate in the applications of response surface methods. A catalog of multi-block designs, limited to those particularly applicable to response surface methods, was given by Holms (1967). Response surface methodology assumes that hypercube and star blocks will contain "center points." Criteria for the numbers of such points to use, together with tables of recommended numbers, were given by Box and Hunter (1957). Criteria leading to much smaller numbers of center points for single block experiments were given by Lucas (1976). The purpose of Holms (1979) was to characterize the experiment designer's options for numbers of center points in a range from very small to moderately large for multiblock sequential designs. The multiblock sequential designs were those for which treatment tables had been given by Holms (1967). The numbers of center points used in each of the hypercube blocks ranged from zero to six.

The purpose of the present investigation is to identify chain pooling type sequential deletion procedures that will minimize the prediction errors in models fitted to the results of experiment designs having 16 hypercube points and any of zero to six center points. The investigation used Monte Carlo studies, and the results are exhibited as tables giving some of the operating characteristics of admissible strategies for each of the center point options. A security regret strategy is identified within each set of admissible strategies, and it is shown to be useful for a wide range of coefficients of variation.

MULTISTAGE DECISION PROCEDURE

Population Model

The single observation value of the response is assumed to occur according to the model

$$y = E(Y) + e \tag{1}$$

where e is independently normally distributed with mean zero and variance σ^2. (Some robustness against nonnormality for a chain pooling procedure was demonstrated by Holms and Berrettoni (1967).)

For relatively saturated experiments that are smaller than 16 observations, the opinion is offered that such experiments are too small to provide both (1) good estimates of model coefficients and (2) a good test statistic, in cases where random errors are large enough to call for a statistical decision procedure. The simulations of the present investigation were all performed with experiments containing 16 hypercube points plus zero to six center points in the belief that such experiments are large enough to justify the use of a statistical decision procedure, but small enough so that the precise optimization of the procedure would be quite beneficial. Where g is the number of independent variables, and the experiment is assumed to be a 2^{-h} fractional replicate of the full factorial experiment, the factorial observations are assumed to result one-for-one from the hypercube points and their numer is

$$n_c = 2^{g-h} = 16$$

An example of an appropriate model equation for the population mean value of the response in the case of four independent variables is

$$\begin{aligned} E(Y) = {} & \beta_1 + \beta_2 x_1 + \beta_3 x_2 + \beta_4 x_1 x_2 + \beta_5 x_3 + \beta_6 x_1 x_3 + \beta_7 x_2 x_3 \\ & + \beta_8 x_1 x_2 x_3 + \beta_9 x_4 + \beta_{10} x_1 x_4 + \beta_{11} x_2 x_4 + \beta_{12} x_1 x_2 x_4 \\ & + \beta_{13} x_3 x_4 + \beta_{14} x_1 x_3 x_4 + \beta_{15} x_2 x_3 x_4 + \beta_{16} x_1 x_2 x_3 x_4 \end{aligned} \tag{2}$$

The model is assumed linear with orthogonality of parameter estimates provided by the design of the experiment or by an orthogonalizing transformation (Holms (1974)). The subsequent discussion assumes that an equation such as (2) will

be fitted to the results of a two-level experiment where the x's are "design values," namely, the high level of x_k is represented by $x_k = +1$ and the low level of x_k is represented by $x_k = -1$. (If center points are used, they have coordinates with all $x_k = 0$).

The initial model fitting is assumed to give least squares estimates of the model parameters that are minimum variance unbiassed linear estimates and for parameters beyond β_1 have the form

$$b_i = \frac{1}{n_c} \sum_{k=1}^{n_c} a_{ik} y_k \tag{3}$$

where $i = 2, \ldots, n_c$ and the a_{ik} are appropriate values of plus or minus one. Such estimates have expectations

$$E(b_i) = \beta_i \tag{4}$$

Combination Estimate For Zero Degree Coefficient

A weighted estimate of the β_1 of equation (2) is to be formed from the n_c hypercube observations and the n_0 center point observations where all observations are assumed to have variance σ^2. Model coefficients estimated from the hypercube observations each have variance

$$V(b_i) = \sigma^2/n_c$$

Thus, the variance of the function estimate for a model such as equation (2) with coefficients all estimated, for example, by Yates' method from such observations is (at the design center)

$$V(\hat{Y}_0) = V(b_1) = \sigma^2/n_c$$

Let Y_0 be estimated from a combination of the n_c hypercube points and the n_0 center points. Let $\bar{y}_0$ be the arithmetic mean of the n_0 center point observations. Then the estimate of Y_0 weighted inversely as the variances of b_1 and $\bar{y}_0$ is

$$\hat{Y}_0 = (n_c b_1 + n_0 \bar{y}_0)/(n_c + n_0)$$

Because the coefficient estimates are uncorrelated, the weighted estimate $\hat{Y}_0$ is also the least squares estimate of β_1. Thus, if b_1 is the estimate of the zero degree coefficient from the Yates analysis, the least squares estimate from the combined observations is

$$b_1^* = (n_c b_1 + n_0 \bar{y}_0)/(n_c + n_0) \tag{5}$$

Mean Squares and Sums of Squares

The squares of the estimates multiplied by n_c provide the numerator mean squares used in the hypothesis testing.

$$z_i = n_c b_i^2 \tag{6}$$

These mean squares have expectations

$$E(Z_i) = \sigma^2 + n_c \beta_i^2 \tag{7}$$

where

$$V(Y_k) = \sigma^2 \tag{8}$$

for

$$i = 2, \ldots, n_c$$

and for

$$k = 1, \ldots, n_c$$

Thus, from equation (7) if any β_i is zero, the associated Z_i is an estimator of σ^2.

The denominator for the hypothesis testing is based on the construction of sums of squares. Six cases are identified according to combinations of the values of n_0 and m_p where n_0 is determined by the design of the experiment and m_p is chosen according to a strategy of hypothesis testing. The cases are identified by the first three columns of table 1. Equations for the initial sum of squares, SS_0, to be used in the starting denominator of the test statistic are derived in appendix B and given in the fifth column of table 1.

Sequential Deletion

Because case a provides no denominator sum of squares, there can be no deletion procedure. All the model coefficients are estimated and all the terms are retained.

Case d uses $n_0 = 0$ and $m_p > 0$. This is the case investigated by Holms (1969). The deletion method of that investigation was as follows:

The mean squares, Z_i, from the usual Yates' analysis (aside from the zero degree coefficient) are ordered in nondecreasing magnitude and renamed $Z_{(j)}$:

$$Z_{(1)} \leq \cdots \leq Z_{(n)}$$

where $j = 1, \ldots, n$ and $n = n_c - 1$. As stated by Birnbaum (1959) the optimal decision procedure when all except possibly one of the coefficients of an equation such as (2) are zero uses a test developed by Cochran (1941). The statistic is:

$$C_n = Z_{(n)}/(Z_{(1)} + \cdots + Z_{(n)})$$

Chain-pooling assumes that m_p of the smallest $Z_{(j)}$ have been generated with zero population coefficients. Their sum is called R_{j-1} where initially

$$j - 1 = m_p$$

Multiplication of the critical points of Cochran's distribution by j gives the critical points of the U_j distribution tabulated by Holms and Berrettoni (1967). The mean square $Z_{(j)}$ is tested for significance at nominal preliminary level α_p using the statistic

$$U_j = jZ_{(j)}/(R_{j-1} + Z_{(j)}) \tag{9}$$

If $Z_{(j)}$ is not significant, j is indexed upward by one and the next mean square is tested. If any mean square so tested is significant, (e.g., the j-th) then each subsequent larger mean square is tested at a nominal final level α_f, where $\alpha_f \leq \alpha_p$. For example, the k-th mean square is tested using the statistic

$$U_j = jZ_k/(R_{j-1} + Z_k) \tag{10}$$

If the k-th ordered mean square is the smallest mean square to test significant at level α_f then all terms associated with smaller mean squares are deleted from the model. Because the assumptions of Cochran's distribution are thereby repeatedly violated, the useful values of the strategy parameters (m_p, α_p, α_f) were determined from simulation studies.

The generalization of the strategy (m_p, α_p, α_f) investigated by Holms (1977b) included a sequence of analysis cycles, but showed that merely one cycle was sufficient. The cases with $n_0 > 0$ are cases b, c, e, and f. A hypothesis testing procedure more general than Holms (1977b) is appropriate for these cases.

Consider cases b and c where $n_0 > 0$ and $m_p = 0$. The first mean square to be tested is $Z_{(1)}$ and the null hypothesis is

$$H_0: \ \beta_{(1)} = 0$$

where for any j, $\beta_{(j)}$ is the parameter associated with the ordered mean square $Z_{(j)}$. The alternative hypothesis is

$$H_a: \ \beta_{(1)} > 0$$

Because the U_j is not defined for $j < 2$, $Z_{(1)}$ cannot be tested against the U_j distribution. If the test is performed against the F-distribution, the fact that $Z_{(1)}$ is an ordered statistic implies that a test of nominal size α will not have true size α. With this proviso, a nominal size α-test is performed.

For case b the test statistic is

$$F_{1,ndfb} = \frac{n_{dfb} Z_{(1)}}{SS_b} \tag{11}$$

with n_{dfb} defined by equation (B3) and with SS_b computed by equation (B4).

For case c the equivalent test statistic is

$$F_{1,ndfc} = \frac{n_{dfc} Z_{(1)}}{SS_c} \tag{12}$$

where n_{dfc} is given by equation (B6) and SS_c is given by equation (B5).

If $Z_{(1)}$ is reported significant then no further testing is done and there is no conditional deletion of terms.

For either of cases b or c let SS_0 be the initial sum of squares and let n_{df0} be the initial degrees of freedom. If $Z_{(1)}$ was reported as insignificant, then it is pooled with SS_0 for a test of $Z_{(2)}$. The test statistic is then

$$F_{1,ndf0+1} = \frac{(n_{df0} + 1) Z_{(2)}}{SS_0 + Z_{(1)}} \tag{13}$$

Testing and pooling continue in this manner, provided insignificance is the result of the prior test.

The test statistic for any $Z_{(j+1)}$ is thus

$$F_{1,ndf0+j} = \frac{(n_{df0} + j) Z_{(j+1)}}{SS_0 + Z_{(1)} + \cdots + Z_{(j)}} \tag{14}$$

For $j > 1$ the option exists of testing $Z_{(j)}$ against the F-distribution or against the U_j-distribution. These options are also both available for the first test of a $Z_{(j)}$ in cases (d), (e), and (f), however, testing against the F-distribution might not be good for case (d), because in case (d), there is neither a pure error nor a residual sum of squares, and the testing is performed entirely with ordered mean squares.

Suppose the situation is that of $n_0 > 0$ and $j > 1$. A criterion is needed for choosing between testing against the F-distribution or against the U_j- distribution. If j is relatively small and n_0 is relatively large, the F-distribution might be more appropriate, whereas if j is relatively large and n_0 is relatively small, the U_j-distribution might be more appropriate. One approach could be, for $j > 1$, to compute j/n_0 and use the F-distribution for $j/n_0 \leq r_F$ and use the U_j-distribution for $j/n_0 > r_F$ where $0 \leq r_F < \infty$ and where r_F has been optimized from Monte Carlo studies. Table 2 shows how the choice of the value of r_F affects the values of n_0 and j at which a transfer occurs from the use of the F-distribution to the use of the U_j-distribution.

Consider the case

$$j/n_0 > r_F \tag{15}$$

The U_j statistic is defined by equation (9). Suppose the criterion $j > r_F n_0$

has been met and the information in $y_{01}, \cdots, y_{0n0}$ is to be combined with that in $Z_{(1)}, \cdots, Z_{(j)}$ for a test of $Z_{(j)}$ against a critical point of the U_j-distribution. An approximation to equation (9) is

$$t_j = \frac{(n_{df0} + j)z_{(j)}}{SS_0 + z_{(1)} + \cdots + z_{(j)}} \tag{16}$$

The distribution of t_j of equation (16) is merely an approximation to the distribution of U_j of equation (9) because the denominator of equation (16) has been stabilized by the n_{df0} mean squares in SS_0.

Under the null hypothesis, $z_{(j)}$ is the largest of an ordered sample of j estimators of σ^2. Although the quantity $(SS_0 + z_{(1)} + \cdots + z_{(j)})/(n_{df0} + j)$ is an estimator of σ^2 as is the quantity $(r_{j-1} + z_{(j)})/j$ of equation (9) the quantity j of equation (16) rather than the quantity $(n_{df0} + j)$ was used as the entry point of the U_j-tables, because the tables are based in part on the numerator $z_{(j)}$ being the j-th extreme value of a sample of mean squares together having mean value σ^2.

For case d, use of the U_j-distribution implies that the first test using equation (9) takes place with

$$r_{j-1} = SS_d$$

where SS_d is defined by equation (B7) and $j = m_p + 1$. Subsequent testing is done with

$$t_j = \frac{jz_{(j)}}{SS_d + z_{(m_p+1)} + \cdots + z_{(j)}}$$

For either of cases e or f, with $j \leq r_F n_0$, the test statistic is provided by equation (14) with SS_0 and n_{df0} being given by equations (B9) and (B10) for case e and by (B11) and (B12) for case f. If $Z_{(j)}$ tests as insignificant, then it is pooled with (added to) the denominator of the test statistic and j is indexed upward one unit. When some $Z_{(j)}$ tests as significant, the testing is stopped and $\hat{n}$ terms are deleted from the model where $\hat{n}$ is the integer value of $r_\eta(j - 1)$ where r_η has been empirically optimized. The terms deleted are those corresponding to the $\hat{n}$ smallest mean squares. If $Z_{(j)}$ tests as insignificant at $j = n_c - 1$, then $\hat{n}$ is the integer value of $r_\eta(n_c - 1)$.

Definition of Strategy

In summary, the expressions for degrees of freedom and sums of squares are given in table 1, and the entire sequential deletion strategy is specified by the parameters $(m_p, r_F, \alpha_F, \alpha_U, r_\eta)$. Where m_p has range $0 \leq m_p < n_c$, m_p is the number of mean squares initially pooled. If $Z_{(j)}$ is the j-th ordered mean square being tested, then $Z_{(j)}$ is tested against the U_j- distribution at nominal level α_U if $j > 1$ and $j > r_F n_0$. The mean square $Z_{(j)}$ is otherwise tested against the F-distribution at nominal level α_F. The convention $\alpha_F = 1.0$ is used to signify that no testing is done against the F-distribution, and the convention $\alpha_U = 1.0$ is used to signify that no testing is done against the U_j-distribution. The number of terms found to be insignificant is

multiplied by r_η and the integer value of the product, namely, $\hat{\eta}$, is the number of smallest absolute value coefficients whose terms will be deleted from the model. (The coefficient, b_1^*, of the zero degree term is excluded from the deletion procedure.) In the notation for the strategy parameter set (m_p, r_F, α_F, α_U, r_η), a long dash will be used to represent a parameter if it has been made inoperative by a value assigned to some other parameter.

SIMULATION PROCEDURE

Unfavorable Population Model

The basic chain pooling concept was investigated for the purpose of minimizing prediction error as described by Holms (1974). That investigation was concerned with the fitting of models to fractional factorial experiments under the condition of population functions of irregular shape. The present emphasis is on the condition where the relative values of the population coefficients are all unfavorable to the deletion procedure. For reasons given by Holms and Berrettoni (1969) this condition is achieved by proportioning the squares of the δ_k values to the expected values of the order statistics of a single $\chi^2_{(1)}$ distribution.

To accomplish such proportioning, let ζ_k be the expectation of the k-th order statistic among ρ independent $\chi^2_{(1)}$ statistics. Let

$$\delta_k^2 = \zeta_{\rho-k+1} \qquad \text{for} \qquad k = 1, \cdots, \rho$$

Expectations of order statistics from a gamma distribution with scale parameter one, shape parameter 1/2 and many sample sizes have been tabulated (Harter, H. L. (1964)). Multiplying such values by 2 gives the expectations of the order statistics of the central $\chi^2_{(1)}$ distribution. Such expectations, for a sample of size ρ, provide the values called for by the definition of the ζ_k.

Equation (13) of Holms and Berrettoni (1969) now gives for the coefficients:

$$\beta_i = \left(\frac{\lambda\sigma^2}{2^{g-h}}\right)^{1/2} \delta_k \tag{17}$$

where $k = 1, 2, 3, \cdots, \rho$; $i = 1, 2, \cdots, n_c$ and $i = k + 1$ for $k = 1, \cdots, \rho$; and $\beta_i = 0$ otherwise.

The statistician's strategy consists of the components (m_p, r_F, α_F, α_U, r_η). Nature's strategy consists of the number, ρ, of non-null population parameters, and the mean noncentrality parameter, λ. Let

$$\theta^2 = \frac{\lambda}{2^{g-h}} \tag{18}$$

From equations (17) and (18)

$$\beta_i = \theta\sigma\delta_k \tag{19}$$

Because of the freedom to choose the value λ and because λ is a scale parameter on σ^2 (eq. (17)) an investigation of the effect of variations in σ^2 is superfluous, and σ^2 will be set equal to one.

In general, the smaller the number of null mean squares, η, the greater will be the probability of decision errors. This was illustrated in figure 4 of the paper by Holms and Berrettoni (1969). Thus, the most difficult situation arises for $\eta = 0$. For the 2^{g-h} experiment with $g - h = 4$, and where the β_1 term (zero degree term) is not subjected to testing, the condition equivalent to $\eta = 0$ is the condition $\rho = 15$.

As developed by Holms (1977b), a nature's strategy with $\rho = 15$ and a normal distribution of model parameters would seem to be a highly likely strategy, and correspondingly, a statistician's strategy optimized against such a nature's strategy should be thought of as a Bayes strategy. As developed by Holms and Berrettoni (1969), such a normal distribution of model parameters is represented by the parameter distributions of table 3 and these distributions are highly unfavorable to the statistical decision procedure. A procedure optimized against $\rho = 15$ and the distribution of δ_k of table 3 may therefore also be regarded as a security strategy. Such a nature's strategy (table 3) will therefore be chosen as the strategy against which the statistician's strategy will be optimized, and such optimization will therefore combine the Bayes and security attributes.

Steps of Simulations

Ordinarily, in the analysis of a real experiment, Yates' method would be applied to the observations to give estimates of the model coefficients. The population mean values and the errors of the observations would be unknown, but in this investigation, the values of the population mean values are required to be known. The steps in a simulated experiment were as follows:

1. An unfavorable set of β_i were constructed as indicated by equation (19).

2. Population mean values, μ_i, for the simulated observations were computed from the β_i using the reversed Yates' method of Duckworth (1965).

3. Pseudo normal random errors, e_i, were generated as described by Holms (1977b).

4. The simulated observations, y_i, were generated by

$$y_i = \mu_i + e_i \tag{20}$$

5. The b_i were estimated from the y_i using Yates' method, except for b_1^*, which weighted in the y_{0k} as in the preceding section (eq. (5)).

6. Some of the b_i were set equal to zero using the strategy

$$(m_p, r_F, \alpha_F, \alpha_U, r_\lambda)$$

7. The reversed Yates' method of Duckworth (1965) was used to compute predicted values, $\hat{y}_i$, using the reduced set of b_i.

8. The prediction errors were computed from

$$e_{pi} = \hat{y}_i - \mu_i \qquad i = 1,2,\cdots,16 \tag{21}$$

Additional details of these steps are given in appendices C and D.

Magnitude of Scale (Noncentrality) Parameter

Any particular strategy (m_p, r_F, α_F, α_U, r_η) was evaluated for an array of populations having five unique values of the mean noncentrality parameter, λ, namely, 0.25, 1.00, 4.00, 16.00, and 64.00. From equation (18) with $n_c = 2^{g-h} = 16$ the corresponding values of θ are 0.125, 0.25, 0.5, 1.0, and 2.0.

As developed by Holms (1977b), the reduction of $V(\hat{Y}_i)$ achievable by deleting terms is σ^2/n_c for each term deleted. On the other hand, if equation (2) is the population model, and if the x-values are all +1, the bias in $\hat{Y}$ is increased by the amount of β_i for each β_i value that is deleted. Thus, an optimal strategy to minimize the squared error of $\hat{Y}$ should not only delete all terms for which the population β_i is zero, it should also delete at least all terms for which the bias contribution to mean square error is less than the variance contribution.

Because in the simulations all $\beta_i \geqq 0$ and as indicated by equation (2), for that point of the experiment where all of the x-values have the value +1, the expected value of Y_i takes on its greatest absolute value, which is (eq. (19))

$$\mu_{max} = \max_i \left[E(Y_i)\right] = \sum_{j=1}^{\rho} \theta\delta_j$$

The values of $\sum_{j=1}^{\rho} \theta\delta_j$ for $\rho = 15$ have been listed in table 3. Because $\sigma = 1$, these values are also the values of μ_{max}/σ.

Reciprocals of μ_{max}/σ are here defined as coefficients of variation for the maximum population mean values. From table 3 such coefficients vary from a high of 64.3 percent (at $\theta = 0.125$) to a low of 4.0 percent (at $\theta = 2.000$). This range of such a coefficient of variation suggests that the range of $0.125 \leq \theta \leq 2.000$ is an adequately wide range of θ to represent the situations that an experimenter might encounter.

EVALUATION CRITERIA

Where e_{0in} are the "observation" errors, namely, the pseudo normal random numbers generated in the nth simulation, the "observations" are given consistently with equation (20) by

$$y_{0i\ell n} = \mu_{i\ell} + e_{0in} \tag{22}$$

$$\ell = 1,2,\ldots,\ell_\theta;\ i = 1,2,\ldots,n_c$$

Following the selection of terms (where some of the coefficient estimates are set equal to zero), the predicted values of the dependent variable are computed for all the hypercube combinations of the independent variables, by the reversed Yates' method of Duckworth (1965). The difference between predicted values, $y_{pi\ell mn}$, of the dependent variable for the nth simulation and the population mean will be called the prediction error, and thus it is (consistent with eq. (21)):

$$e_{pi\ell mn} = y_{pi\ell mn} - \mu_{i\ell} \tag{23}$$

$$\ell = 1,2,\ldots,\ell_\theta;\ m = 1,2,\ldots,n_{n0};\ i = 1,2,\ldots,n_c$$

Over the n_e simulations, the sample mean square error of prediction for a given treatment is

$$\bar{e}^2_{pi\ell m} = \frac{1}{n_e}\sum_{n=1}^{n_e} e^2_{pi\ell mn} \tag{24}$$

The maximum of such errors over the treatments is

$$\bar{e}^2_{max} = \bar{e}^2_{p\ell m,max} = \max_{i = 1,\ldots,n_c} (\bar{e}^2_{pi\ell m}) \tag{25}$$

The mean of the squared error over the simulations and over the points of the space of the experiment is

$$\bar{e}^2_{p\ell m} = \frac{1}{n_c}\sum_{i=1}^{n_c} \bar{e}^2_{pi\ell m} \tag{26}$$

Equations (25) and (26) provide two criteria for measuring the effectiveness of a strategy. The particular set of values of strategy parameters that minimizes $\bar{e}^2_{p\ell m,max}$ (as given by eq. (25)) can be called a security strategy, and if the points of the space of the experiment are assumed to be equally likely of being of interest, the particular set that minimizes $\bar{e}^2_{p\ell m}$ can be called an approximate Bayes strategy. For either criterion, the values of squared errors would have been the prime consideration.

The criteria of equations (25) and (26) were evaluated using computer simulations using 1000 experiments. Thus, the long run mean squared error of the decision procedures was evaluated. This leaves open the question of how badly a decision procedure might perform in individual cases. One approach to this question is to evaluate the stability of the mean squared errors observed in the simulations. Thus, in addition to the criteria of equations (25) and (26) two other criteria for the effectiveness of a strategy were investigated. They are concerned with the stability of the quantities defined by equations (25) and (26). The instability of these criteria can be measured by the variance of the square of the prediction error. The estimate of the variance of $e^2_{pi\ell mn}$ is

$$\hat{V}\left(e^2_{pi\ell m}\right) = \frac{1}{n_e - 1}\left[\sum_{n=1}^{n_e}\left(e^2_{pi\ell mn}\right)^2 - \frac{1}{n_e}\left(\sum_{n=1}^{n_e} e^2_{pi\ell mn}\right)^2\right] \tag{27}$$

Equation (27) gives an unbiased estimate of the variance of the squared error over n_e simulations. The maximum of this quantity over the space of the simulated experiments is defined by

$$V\left(e^2\right)_{max} = \hat{V}\left(e^2_{\ell m}\right)_{max} = \max_{i=1,\ldots,n_c}\left[\hat{V}\left(e^2_{pi\ell m}\right)\right] \tag{28}$$

The arithmetic mean of the variance of the squared error over the space of the experiments is defined by

$$\hat{V}\left(\overline{e}^2_{\ell m}\right) = \frac{1}{n_c}\sum_{i=1}^{n_c}\hat{V}\left(e^2_{pi\ell m}\right) \tag{29}$$

The average number of terms, $\overline{p}_{\ell m}$, selected by the strategy, is computed for each of the values of θ_ℓ, $\ell = 1, \ldots, \ell_\theta$ and for each of the values of n_0, $m = 1,\ldots,n_{n0}$. The program also computes the ratio of the maximum prediction error to the scale parameter θ. The ratio is computed from θ_ℓ and from the $\overline{e}_{max}$ of equation (25):

$$C_{ee,mx}(\theta) = \frac{\overline{e}_{p\ell m,max}}{\theta_\ell} \tag{30}$$

The value of $C_{ee,mx}(\theta)$ of the preceding equation was adjusted to penalize it for the increased experimentation needed for the reduction in variance that might be expected from the additional center point observations. Thus, with

$$n_t = n_c + n_0$$

$$C_{ae,mx}(\theta) = n_t^{1/2} C_{ee,mx}(\theta) \tag{31}$$

Although the experiments were simulated, the model fitting and selection was performed, and predicted values were computed as if the experiments were full factorial experiments, the conclusions ot the investigation are not necessarily limited to full factorial experiments. The errors of the predicted values were always evaluated at points of the space of the experiment for which "observations" were available. Thus, the conclusions of the experiment are equally applicable to regular fractional factorial experiments with 16 treatments, provided that the only concern is with prediction errors at the points of the experiment where observations were actually acquired. Thus, for example, if the experiment were a one quarter replicate on 6 independent variables, the strategy recommendations apply to predictions for the 16 hypercube conditions actually performed. The errors might be much larger, and a different sequential deletion strategy might be preferred, if predictions were to be made for some of the 48 treatments that had not been performed. As shown by Holms (1974), such predictions should be based on a far more stringent deletion strategy than for the case of predictions limited to points of actual observations.

COMPUTER PROGRAM

Computations were performed using the computer program, POOL9U. Details of the program are given in appendices C, D, and E. The manner of repeated use of arrays for the simulated observations and estimated model parameters is shown by figure 1. The major program logic is exhibited by figure 2. The branch points for the computation of sums of squares according to the six cases of table 1 are exhibited by figure 2(a). The procedure for the significance tests is exhibited by figure 2(b), and the final deletion procedure is exhibited by figure 2(c).

SIMULATION RESULTS

Results of an investigation of the effect of Monte Carlo sample size on the stability of the empirical results for a similar chain pooling strategy were given by Holms (1977b). In general, the results converged to a constant when the number of sampled experiments was 1000 or more. Variability of results occurred as the number of sampled experiments was reduced below 1000. All of the strategy comparisons of the present investigation were performed for 1000 sampled experiments. All simulations were performed for $\rho = 15$. The strategies were compared in terms of the maximum coefficient of error, $C_{ae,mx}$, adjusted for n_0, as defined at equation (31).

Large Coefficient of Variation

The investigation of Holms (1977b) concerned the case of $n_0 = 0$. One of the conclusions was that if the investigator has prior knowledge that the relative error is quite large (coefficients of variation in the neighborhood of 65 percent), the strategy should immediately delete the five smallest absolute value terms and then test with continued pooling at a nominal test level of 0.05 against the U_j-distribution, to estimate a number $\hat{\eta}$ of insignificant terms. The optimum number of terms deleted from the model was shown to be the integer value of $1 + 0.675\ \hat{\eta}$.

The strategy parameters of the present investigation are (m_p, r_F, α_F, α_U, and r_η). With $n_0 = 0$, no testing or deletion can be accomplished unless $m_p > 0$. As exhibited by figure 2(b), the initial value of j is $m_p + 1$. Thus, even if $\alpha_F < 1.0$, figure 2(b) shows that with $m_p > 0$, j is > 1, and with $n_0 = 0$, j is $> r_F n_0$ for any finite r_F, and thus control is transferred to statement 418, and testing against the F-distribution is excluded. Thus, any testing with $n_0 = 0$ is done against the U_j-distribution.

In the present investigation, the number of terms deleted from the model is $r_\eta \hat{n}$. In Holms (1977b) the number was $1 + r_2\hat{n}$. Thus, for an equal number of terms to be deleted in the two investigations,

$$r_\eta \hat{n} = (1 + r_2)\hat{n}$$

from which

$$r_\eta = r_2 + 1/\hat{n}$$

Thus, whereas $r_2 = 0.675$ was found to be optimum for large coefficients of variation ($\theta = 0.125$) in Holms (1977b), a value of r_η somewhat larger than 0.675 should be anticipated to be optimum for $n_0 = 0$ and $\theta = 0.125$ in the present investigation.

From the preceding discussion, an optimum strategy for $n_0 = 0$ and $\theta = 0.125$ should be anticipated to occur in the domain extending to larger values of r_η beginning with the strategy (m_p, r_F, α_F, α_U, r_η) = (5, ——, 1.00, 0.05, 0.675). ($\alpha_F = 1.00$ makes r_F inoperative in the preceding discussion.) This anticipation was confirmed in that the best strategy for $n_0 = 0$ and $\theta = 0.125$ was (5, ——, 1.0, 0.05, 0.75).

The best strategies for $\theta = 0.125$ and for each of $n_0 = 0$, 1, 2, 3, 4, 5, and 6 are listed in the last row of table 4 for each value of n_0.

Small Coefficient of Variation

If the statistician's loss function is the maximum adjusted relative error over the space of the experiment, $C_{ae,mx}$, then the strategy that is optimal for $\theta = 0.125$ is a security strategy because within the present investigation $C_{ae,mx}$ is larger for $\theta = 0.125$ than for any other value of θ investigated. On the other hand, if the statistician's loss function is simply the absolute value of the maximum squared error over the space of the experiment, namely, $\bar{e}^2_{max}$ as defined by equation (25) then (with any sequential deletion) that quantity is a maximum within the present investigation at $\theta = 2.000$ and thus the security strategy for such a loss function would be the strategy that minimizes $C_{ae,max}$ (2.000).

The strategy anticipated to minimize $C_{ae,mx}$ (2.0) is the strategy with no deletion, which is symbolized as (m_p, r_F, α_F, α_U, r_η) = (0, ——, 1.0, 1.0, 0.0), which results in $\bar{\nu} = 15$. This anticipation was realized for $n_0 = 0$, but for the larger values of n_0, some deletion (resulting in $\bar{\nu} < 15.0$) actually gave the best strategies for $\theta = 2.0$. (Results and operating characteristics of the strategies that gave the smallest observed values of $C_{ae,mx}$ (2.0) are listed in the first row of table 4, for each value of n_0.)

Admissible Strategies

For the purposes of the present investigation, a strategy will be classed either as admissible or as dominated according to its values of $C_{ae,mx}(\theta)$ at both $\theta = 0.125$ and $\theta = 2.000$. A strategy will be said to be dominated if for $\theta = 0.125$ there is another strategy with the same or lesser $C_{ae,mx}$ (0.125) and with a lesser $C_{ae,mx}$ (2.000). A strategy will also be said to be dominated if there is another strategy with the same or lesser $C_{ae,mx}$ (2.000) and with a lesser $C_{ae,mx}$ (0.125).

Any strategy that is not dominated is defined as being admissible. The strategies found to be admissible are listed in table 4, together with some of their operating characteristics.

Security Regret Strategies

The strategies of table 4 have been listed for each value of n_0 in the nondecreasing order of $C_{ae,mx}$ (2.0). Thus, the first strategy listed for each n_0 is the strategy giving the smallest value of $C_{ae,mx}$ (2.0) for the given n_0. The last strategy listed in table 4 for any given n_0 is a strategy giving the smallest value of $C_{ae,mx}$ (0.125).

The regret function of a statistical decision procedure, as a function of a parameter θ, is here defined as the excess loss occurring with the procedure at a particular value of θ as compared with the loss that would have occurred had the best statistical decision procedure been used for that particular value of θ. For the purposes of the present investigation a regret function $R(\theta)$ is defined for $\theta = 0.125$ as being the $C_{ae,mx}$ (0.125) for any strategy divided by the value of $C_{ae,mx}$ for the best strategy for that value of θ, and $R(\theta)$ is defined for $\theta = 2.000$ as being the $C_{ae,mx}$ (2.000) for any strategy divided by the value of $C_{ae,mx}$ for the best strategy for that value of θ.

Thus, for the successive values of n_0, the regret functions $R(n_0, \theta)$ are

$$R(n_0, 0.125) = C_{ae,mx}(n_0, 0.125)/\min\left[C_{ae,mx}(n_0, 0.125)\right]$$

and

$$R(n_0, 2.0) = C_{ae,mx}(n_0, 2.0)/\min\left[C_{ae,mx}(n_0, 2.0)\right]$$

From table 4, the values of

$$\min\left[C_{ae,mx}(n_0, 0.125)\right] \qquad \text{and} \qquad \min\left[C_{ae,mx}(n_0, 2.0)\right]$$

are as follows,

n_0	$\min\left[C_{ae,mx}(n_0, 0.125)\right]$	$\min\left[C_{ae,mx}(n_0, 2.0)\right]$
0	29.46	2.065
1	30.39	2.118
2	31.11	2.187
3	31.95	2.244
4	32.63	2.300
5	33.51	2.352
6	34.25	2.401

The <u>single strategy</u> that has the smallest regret function over both $\theta = 0.125$ and $\theta = 2.0$ is defined as the <u>security regret strategy</u>. The security regret strategy is thus the sequential deletion procedure, which for a given n_0, produces the least increase in prediction error for $\rho = 15$ and an unfavorable distribution of parameters over that prediction error which could have been achieved if the best strategy had been chosen for the given (unknown) value of error variance, σ^2.

In examining the $R(\theta)$ values of table 4 for a given value of n_0, the parameters that give the security regret strategies are those that give the joint minimums on $R(0.125)$ and $R(2.0)$, and these joint minimums have been identified by asterisks. Thus, for the given values of n_0, the security regret strategies and the associated values of $C_{ae,mx}(\theta)$ are as follows,

n_0	m_p	r_F	α_F	α_U	r_η	$C_{ae,mx}$ (0.125)	$C_{ae,mx}$ (2.0)
0	1	---	1.0	0.50	0.25	32.95	2.240
1	0	3.0	.50	.10	.80	31.78	2.180
2	0	3.0	.25	.50	.85	33.14	2.252
3	0	0.0	.75	.50	.80	33.29	2.301
4	0	0.5	.50	.50	.80	33.89	2.309
5	0	1.0	.25	.10	.80	33.97	2.394
6	0	0.5	.50	.05	.80	34.72	2.408

The question can be asked as to what choice of n_0 will result in the most efficient experiment. If the object of a choice of n_0 is to use the most efficient choice together with a security regret strategy for deleting terms, then the preceding table shows that the most efficient choice (the choice that minimizes each of $C_{ae,mx}$ (0.125) and $C_{ae,mx}$ (2.0)) is the choice of $n_0 = 1$. This choice applies to the condition of $n_c = 16$.

Selection of a Strategy

In summary, if the experimenter wishes to minimize the maximum prediction error over the 16 hypercube points of an experiment with n_0 center points when the variance error is relatively large (coefficient of variation in the range of 65 percent), the strategy for a given n_0 should be the last listed strategy (for the given n_0) of table 4. If the experimenter wishes to minimize the maximum prediction error over the points of the experiment when the variance error is relatively small (coefficient of variation in the range of 4 percent) the strategy for a given n_0 should be the first listed strategy (for the given n_0) of table 4.

If the experimenter has no basis for a choice of one of the two preceding extreme choices, the choice should be a security regret strategy as indicated by the asterisked results in table 4, in which case (for all of the n_0 values) the largest value of the regret function will be $R(0.125) = 1.1185$ as listed in table 4(a). This value of the regret function shows that for the worst value of $n_0 (n_0 = 0)$, the relative prediction standard error is increased by at most about 12 percent over what it would have been if the worst value of θ had occurred and the best strategy against it had been used. Thus, the security regret strategies (for each of the values of n_0) must be concluded to be widely useful strategies.

Variance of Predicted Squared Error

The strategy selections described in the preceding section are based on a Monte Carlo investigation that reported mean values of prediction errors over 1000 simulations. The quoted results thus tell what the mean long run results will be as a function of strategy selection. The subject of short run results was not discussed. Some insight into the short run performance can be gained by examining the observed values of $V(e^2)_{mx}$. This quantity gives the observed variance, for samples of size 1000, of the maximum squared prediction errors over the simulations, as defined by equation (28). If this variance is relatively small, then operating characteristics such as $C_{ae,mx}(\theta)$ are relatively constant from simulation to simulation. But, if $V(e^2)_{mx}$ is relatively large, then the short run performance of a strategy could be erratic.

In the case of large coefficients of variation (small values of θ) the strategy performance was not erratic - the values of $V(e^2)_{mx}$ were small for all of the strategies of table 4 for $\theta = 0.125$. The strategy performance can be erratic for small coefficients of variation (large values of θ). Thus, the values of $V(e^2)_{mx}$ were large or small for $\theta = 2.000$, depending on the strategy (table 4). This response to θ shows that the bias component is the component of the prediction error that can be erratic. In particular, the values of $V(e^2)_{mx}$ were large for $\theta = 2.0$ when strategies were used (table 4) that would result in the smaller values of $C_{ae,mx}$ (0.125). Thus, a strategy favorable to large coefficients of varia ion should never be used if the possibility exists that the coefficient of variation might be small. In such a state of prior knowledge, the security regret strategy for the given n_0 should be used because the $V(e^2)_{mx}$ for it (table 4) was never very large.

Expected Number of Terms Retained

Some insight into the operation of the proposed strategies can be gained from an examination of the mean number, $\bar{p}$, of terms retained as a function of n_0, θ, and the choice of strategy. The results for the admissible strategies were given in table 4 and are summarized in table 5. Briefly, the strategies that minimize $C_{ae,mx}(\theta)$ for $\theta = 2.0$ simply retain many terms for both $\theta = 2.0$ and $\theta = 0.125$ unless n_0 is relatively large $(n_0 > 3)$. In a somewhat similar manner, the strategies that minimize $C_{ae,mx}(\theta)$ for $\theta = 0.125$ are also insensitive to θ (the value of $\bar{p}$ remains small for both $\theta = 0.125$ and $\theta = 2.0$ unless n_0 is relatively large $(n_0 > 3)$.

By way of contrast, the security regret strategy results in $\bar{\rho}$ being responsive to θ merely provided $n_0 > 0$. Thus, the results in table 5 tend to confirm the previously described results, namely, that $n_0 = 1$ is an efficient value of n_0, and that the security regret strategy is a widely useful strategy.

CONCLUSIONS

An investigation was conducted to determine what statistical techniques should be used for model fitting to the results of a two-level, fixed-effects, full or fractional-factorial, orthogonal experiment with 16 hypercube treatments and zero to six center points when the population model coefficients have an unfavorable distribution of relative values. Sequential deletion strategies using both the F- and a U_j-distribution and combinations of them were evaluated, using Monte Carlo techniques, under the criterion of minimizing the maximum prediction error, wherever it occurred, among the hypercube points.

Three classes of strategies were identified as being appropriate, depending on the extent of the experimenter's prior knowledge. In almost every case, the choice of the strategy was found to be unique, according to the number of center points. Among the three classes of strategies, a security regret class of strategy was demonstrated as being widely useful, in that over a range of coefficients of variation from 4 to 65 percent, the maximum prediction error was never increased by more than 12 percent over what it would have been if the best strategy had been used for the particular coefficient of variation.

Relative efficiency, when using the security regret strategy, was examined as a function of the number of center points, over the range from zero to six, and was found to be best when the design of the experiment added only one center point to the 16 factorial points.

APPENDIX A

SYMBOLS

Mathematical symbol	FORTRAN name	Description
b_i	B(I)	estimate of β_i
$C_{ae,mx}$	ADCOER	adjusted coefficient of error, eq. (31)
$C_{ee,mx}$	COERMX	ratio of maximum prediction error to scale parameter, eq. (30)
E(. . .)		expectation of . . .
e	RN(I)	single observation random error
$\bar{e}^2_{max}$	ERSQMX	maximum over hypercube of mean square prediction error over simulations
g		number of independent variables
g-h	LGMH	experiment contains 2^{g-h} treatments
h		experiment contains $(1/2)^h$ times number of treatments in full factorial experiment
i,j,k	I,J,K	subscripts
ℓ,m,n	L,M,N	
	KODE	amount of NAMELIST output desired
	KPF	index number for α_F
	KPU	index number for α_U
ℓ_θ	LTH	number of θ values investigated in any computer run
m_p	MP	number of mean squares pooled before testing begins
n_0	NO	number of center points
n_c	NC	number of hypercube points
n_t	NT	total number of observations in one experiment
n_e	NE	number of simulated experiments in any strategy evaluation

r_F	RF	distribution transfer parameter, eq. (15)
r_η	RETA	number of terms deleted is integer value of r_η times number insignificant
V(. . .)		variance of . . .
$V(e^2)_{max}$	VESQMX	maximum over hypercube of sample variance of mean square prediction error over simulations, eq. (28)
x_k		k^{th} independent variable
Y		conceptual value of dependent variable
$\hat{Y}$		estimate of response function from fitted model
y_i	YOBS(I)	observed value of dependent variable
Z_i	Z(I)	mean squares in Yates' order
α_F		nominal significance level of F test
α_U		nominal significance level of U_j test
β_i	B(I)	regression coefficients in Yates' order
δ_k	DELTA(K)	parameter determining relative magnitudes of coefficients in population model, eq. (17)
ξ_j		expectation of j^{th} order statistic of a $\chi^2_{(1)}$ variable
η		number of mean squares having noncentrality parameter of zero
$\hat{\eta}$	ETA	number of mean squares concluded to be null during any analysis
θ_ℓ	THETA(L)	scale parameter
λ		mean over experiment of noncentrality parameters, eq. (17)
λ_i		noncentrality parameter
μ_i	YMU(I,L)	population mean value of Y_i for i^{th} treatment
$\hat{\rho}$	RHO	number of coefficients concluded to be non-null in any simulation

$\bar{\rho}$	AVRHO	mean number of coefficients concluded to be non-null in a strategy investigation
σ		standard deviation of e

APPENDIX B

DERIVATION OF EQUATIONS FOR SUMS OF SQUARES

Case a ($n_0 = 0$, $m_p = 0$). - This case provides no information for a sum of squares for a test statistic.

Case b ($n_0 = 1$, $m_p = 0$). - Let Y_{0k} be the value observed at the k-th center point (origin) observation. Then by the definition of σ^2,

$$V(Y_{0k}) = \sigma^2 \tag{B1}$$

Also, from the definition of σ^2, where Y_i is the i-th hypercube observation,

$$V(Y_i) = \sigma^2 \tag{B2}$$

The object is to estimate σ^2 from the information in the y_i, $i = 1, \ldots, n_c$, and a single center point observation, y_{01}. Because the model coefficient estimates in the two-level fractional factorial experiment are orthogonal, the least squares estimates of the regression coefficients from the combined data are all the same as the Yates estimates, except for the coefficient of the zero degree term, b_1. Its least squares estimate is from equation (5):

$$b_1^* = \left(y_{01} + \sum_{i=1}^{n_c} y_i\right) \Bigg/ (1 + n_c)$$

For any of the treatment points, let Δ_i be the difference between the observed value and the predicted value of Y where Δ_0 is the center point difference and $i = 0, 1, \cdots, n_c$. The Yates' estimate of β_1 is

$$b_1 = \frac{1}{n_c} \sum_{i=1}^{n_c} y_i$$

The predicted values under least squares estimation are therefore all augmented by $b_1^* - b_1$ over their Yates' method predictions.

The differences between the Yates' method predictions and the observations are all zero at the hypercube points, therefore over the $n_c + 1$ treatment points,

$$\Delta_0 = y_{01} - b_1^*$$

$$\Delta_i = b_1 - b_1^* \qquad i = 1, \ldots, n_c$$

The estimate of σ^2 from the residual of the least squares regression is

$$s^2 = \frac{\sum_{i=0}^{n_c} \Delta_i^2}{n_c + 1 - n_c} = (y_{01} - b_1^*)^2 + n_c(b_1 - b_1^*)^2$$

where

$$y_{01} - b_1^* = y_{01} - \frac{y_{01} + \sum_{i=1}^{n_c} y_i}{1 + n_c} = \frac{n_c y_{01} - \sum_{i=1}^{n_c} y_i}{1 + n_c}$$

and

$$b_1 - b_1^* = \frac{\sum_{i=1}^{n_c} y_i}{n_c} - \frac{y_{01} + \sum_{i=1}^{n_c} y_i}{1 + n_c} = \frac{\sum_{i=1}^{n_c} y_i - n_c y_{01}}{n_c(1 + n_c)}$$

Thus,

$$s^2 = (y_{01} - b_1^*)^2 + n_c(b_1 - b_1^*)^2 = \frac{n_c}{1 + n_c} (y_{01} - b_1)^2$$

For $n_c = 16$,

$$s^2 = 0.941176(y_{01} - b_1)^2$$

For this case, the number of degrees of freedom is

$$n_{dfb} = 1 \tag{B3}$$

and the sum of squares is

$$SS_b = s_1^2 n_{dfb} = 0.941176(y_{01} - b_1)^2 \tag{B4}$$

In this case, the error sum of squares was obtained from a residual involving b_1^*. This usage of b_1^* has the disadvantage that it will introduce a bias or "lack of fit" component into the sum of squares if the fitted model is biassed at the center point. Because of this bias risk, a "pure error" sum of squares will be computed if $n_0 > 1$.

Case c ($n_0 > 1$, $m_p = 0$). - This case is treated as follows. Let the center point observations be y_{0k}; $k = 1, \ldots, n_0$. Their sample mean is

$$y_0 = \frac{1}{n_0} \sum_{k=1}^{n_0} y_{0k}$$

and the sum of squares, SS_c for case c is now:

$$SS_c = \sum_{k=1}^{n_0} (y_{0k} - \overline{y}_0)^2 = \sum_{k=1}^{n_0} y_{0k}^2 - \frac{1}{n_0}\left(\sum_{k=1}^{n_0} y_{0k}\right)^2 \tag{B5}$$

where the number of degrees of freedom, n_{dfc} is

$$n_{dfc} = n_0 - 1 \tag{B6}$$

<u>Case d ($n_0 = 0$, $m_p > 0$).</u> - Let

$$SS_d = \sum_{j=1}^{m_p} z_{(j)} \tag{B7}$$

The number of degrees of freedom, n_{dfd} is

$$n_{dfd} = m_p \tag{B8}$$

<u>Case e ($n_0 = 1$, $m_p > 0$).</u> - This is the additive situation of cases b and d:

$$SS_e = SS_b + SS_d \tag{B9}$$

$$n_{dfe} = n_{dfb} + n_{dfd} = 1 + m_p \tag{B10}$$

<u>Case f ($n_0 > 1$, $m_p > 0$).</u> - This case is additive with respect to cases c and d:

$$SS_f = SS_c + SS_d \tag{B11}$$

$$n_{dff} = n_{dfc} + n_{dfd} = n_0 - 1 + m_p \tag{B12}$$

APPENDIX C

DESCRIPTION OF COMPUTER PROGRAM

Computations were performed using the FORTRAN-4 program, POOL9U listed in appendix D. The antecedents of the program were POOL3U (Holms (1966)), POOLMS (Amling and Holms (1973)), POOLES (Holms (1974)) and POOL6U (Holms (1977b)). The program POOL9U is outlined and the parts that are essentially the same as the earlier programs are identified by the section numbers and titles of appendix D in the table that follows. The table is followed by a description of POOL9U. Illustrative output is given in appendix E.

Section number	Section title	Reference program
1A	DECLARATIONS AND TABLES	POOLMS
1B	INPUTS AND CONSTANTS	POOL6U
1C	POPULATION MEANS	POOL6U
1D	STRATEGY	(new)
2	SIMULATIONS AND MODEL FITTING	POOLES
3	CONSTRUCTION AND ORDERING OF MEAN SQUARES	POOLMS
4	DELETION OF TERMS	(new)
5	PREDICTIONS	POOLES
6	ACCUMULATION OF ERRORS	POOLES
7	DETERMINATION OF MAXIMUM AND MEAN SQUARED ERRORS	POOLES
8	OUTPUT	(new)
9	YATES METHOD SUBROUTINE	POOLMS

Section 1A. - Declarations and tables. - The values of the nominal test size α are stored as (ALPHA(I), I = 1,11) and later used as output labels. These values range from 0.001 to 1.0, however, the value of 1.0 obtained by setting the index to 11 is merely a code implying that no significance testing is performed.

The sequential deletion requires critical values gainst which the test statistics are compared. The critical values of F are stored internally as ((FTB(I,J), J = 1,10), I = 1,20) where I indexes on the degrees of freedom and J indexes on the value assigned to α. The critical values of U_j are stored internally as ((TB(I,J), J = 1,10), I = 1,16) where I is the order number in nondecreasing order, and J indexes on the value assigned to α.

Section 1B. - Inputs and constants. - The constants defining the populations, the experiments, and the sequential deletion strategy are read from data cards in the following order, with the order of the fields being the same as the order of the symbols in the following description.

Format	Description
(13A6,A2)	REMARK (I), arbitrary literal information such as particular use of program, date of last change, and so forth.
(3I5)	LGMH, NE, KODE
(I8,5F8.3)	LTH, (THETA(L), L = 1, LTH)
(I4/(10F8.5)	NDELTA, (DELTA(K), K = 1, NDELTA). There are as many (10F8.5) cards as are necessary to read (DELTA(K), K = 1, NDELTA.
(8I2)	NNO, (NO(M), M = 1, NNO)
(3I5,2F5.3)	MP, KPF, KPU, RF, RETA (The associated READ statement is actually in section 1D.)

Section 1C. - Population means. - After the initial constants have been read, the next major operation is the formation of the population mean values. The number of population regression coefficient sets to be examined during the investigation of a strategy is the number, ℓ_θ, of θ-values.

With respect to equation (2) all the population model parameters are first set equal to zero with the DO-loop ending at statement 10. The non-zero values of β_{i+1} are initially set equal to δ_i using the DO-loop ending at statement 20. The DO-loop ending at statement 20 serves the purpose of equation (19) with $\sigma = 1$ and $\theta = 1$. The value of $\sigma = 1$ is retained, but the adjustment for θ is made after the population mean values have been computed.

With the population β-values (aside from θ) established at statement 20, the object is to compute the population mean values from the β-values by the reversed Yates' method (Duckworth (1965)). The first step is to reverse the order of the β-values, which is completed at statement 22. The use of the reversed Yates' method then yields the array YOBS(I) as completed at statement 30. The array YOBS(I) is therefore an array of population means μ_i. This array of population means is to be expanded over the mean noncentrality parameters, λ_ℓ, to give the effect of equation (17). This effect is produced on the population mean values by the multiplication

$$\mu_{i,\ell} = \mu_i\theta_\ell$$

and this operation is completed with the creation of the array YMU(I,L) at statement number 48. The values of $\mu_{i\ell}$ are thus indexed over treatments i, $i = 1, \ldots, n_c$, and over arbitrary values of θ_ℓ; $\ell = 1, \ldots, \ell_\theta$.

The index, i, runs over the mean squares to be analyzed within a single experiment and thus unequal values, δ_i contribute to non-uniform noncentrality parameters within the experiment. The index, ℓ serves to change the scale of

the noncentrality parameter and, therefore, each successive value of ℓ generates a new family of experiments. Changing θ_ℓ thus provides the conditions necessary to investigate the deletion procedures for differing coefficients of variation.

Section 1D. - Strategy. - In terms of mathematical symbols previously defined, the strategy parameters are functions of numbers that are read at statement 50 as follows:

Argument	Function
FORTRAN symbol	Mathematical Symbol
MP	m_p
RF	r_F
KPF	α_F
KPU	α_U
RETA	r_η

More than one model deletion strategy can be evaluated during any computer run. On completion of the evaluation of a particular strategy, control is transferred back to statement 50 for the reading of an additional strategy data card. The operation of the program ends when such cards are exhausted.

The error simulations are generated so that all strategies are compared for the same set of random numbers. This is achieved by reinitializing the random number generator for each new strategy with the statement "CALL SAND(XS)."

The prediction errors and their squares are stored in the arrays ERSQ (I,L,M) and ERSQSQ(I,L,M). These arrays are initially cleared by the loops terminating at statements 97, 98, and 99.

Section 2. - Simulations and model fitting. - The number of experiments simulated is NE. The performance of these experiments and their analysis is controlled by the loop: "DO 699 N = 1, NE." Within each experiment, the random numbers for the $(n_c + n_0)$ "observations" are generated as follows.

The procedure generates a sequence of pseudo random numbers with a rectangular distribution by taking the low order single precision bits of the product r_{r-1}*K where r_{r-1} = previous random number and $r_0 = 1$ and $K = 5^{15}$. This fixed point number is then floated and returned to the calling program as a floating point number between 0 and 1 (Tausky and Todd (1956)).

The rectangular variates are transformed to pseudo-normal variates using a procedure described by Box and Muller (1958). The procedure begins with D_1 and D_2 assumed independent and rectangular on the interval (0,1). In the notation of Box and Muller (1958), the transformations are:

$$X_1 = (-2\ \ell n\ D_1)^{1/2}\cos(2\pi D_2)$$

$$X_2 = (-2\ \ell n\ D_1)^{1/2}\sin(2\pi D_2)$$

The operations are completed at statement number 215.

Each set of random numbers for an experiment is used with all values of the population and design parameters θ and n_0 through the statements "DO 690 L = 1, LTH." For all of these cases, the simulated observation errors, as stored in RN(I), are added to the population mean values (stored in YMU(I,L)) for the particular treatments (I = 1, ...,NC), at statement 224 as required by equation (22). (Beyond n_c an additional n_0 values of RN(I) are used as "center point" observations.)

After synthesizing the "observed" values of YOBS(I) the "SUBROUTINE YATES" (section 9) ending with statement 909 is used to compute the array (B(I) which contains (except for division by the number of treatments) the Yates estimates of the parameters in the manner of equation (3) and in the order of equation (2).

Section 3. - Construction and ordering of mean squares. - The mean squares are formed from the parameter estimates (for those terms beyond β_1) and a pointer function is created within the loop "DO 309, I = 1, NC." As exhibited by figure 1, the array BFM(I) remains intact for $m = 1, \ldots, n_{n0}$, but changes as $\ell = 1, \ldots, \ell_\theta$. (In section 4 the array B(I) will be overwritten for all $m = 1, \ldots, n_{n0}$).

The array of pointers to the B(I) array is created by the statement IND(I) = I. This array will serve to identify the coefficients in the B(I) array after the process of ordering mean squares according to rank. The ordering is done in the sequence of statements ending with 313.

Operations thus far created a column of mean squares Z(J) with mean squares indexed on J in the order of increasing rank, together with a column of integers IND(J) indexed on J. Thus, any address J will lead to a mean square Z(J) and also to the integer IND(J). This integer is the index I that the associated regression coefficient has in the original Yates' order.

The computation of the sums of squares is done for each value of n_0 within the loop: "DO 680 M = 1, NNO." The construction begins following statement 313 and ends with statement 365. The operations are outlined by figure 2(a).

The computation of the sums of squares depends on the values of n_0 and m_p according to cases b, c, d, e, and f of table 1. Three combinations of these cases are identified in the statement immediately preceding statement 320. If $n_0 = 0$, the situation is that of case a or d, and control is transferred to statement 330, following which the SS_d of equation (B7) is evaluated at statement 365.

If $n_0 = 1$, the situation can be that of case b ($n_0 = 1$, $m_p = 0$) or case e ($n_0 = 1$, $m_p > 0$) and the SS_b of equation (B4) is computed at the statement following 325.

If $n_0 > 1$, the situation can be that of case c or case f. The quantity SS_c of equation (B5) is computed at the statement for TEM that follows statement 322.

If m_p and n_0 are each zero, there can be no sequential deletion, and control is transferred to statement 432, and all terms are retained. Setting both α_F and α_U equal to 1.00 (by setting KPF = KPU = 11) is used as a code signifying that no conditional pooling is to be done but that arbitrary deletion is to be accomplished according to values assigned to m_p and r_η. This is done by transferring control to statement 421.

Section 4. - Deletion of terms. - The flow chart for the tests of significance is shown by figure 2(b). The procedure begins at statement 417 and ends at statement 419 (appendix D). The significance tests will have been avoided by earlier statements in section 3 if either $n_0 + m_p = 0$ or both $\alpha_F = 1.0$ and $\alpha_U = 1.0$. Thus, entry at statement 417 requires both $n_0 + m_p > 0$ and at least one of α_F or $\alpha_U < 1.0$. If $\alpha_F = 1.0$, control is transferred to the U_j test which begins at statement 418. If $\alpha_F < 1.0$, control is determined (fig. 2(b)) by the questions: "Is $j > r_F n_0$? and is $j > 1$?" If both are "yes," control is transferred to statement 418, which initiates the U_j-testing.

Irrespective of whether significance testing is against the F-distribution or the U_j-distribution, insignificance pools $Z_{(j)}$ into the denominator of the test statistic and then transfers control to statement 419 which increases j by one unit. Significance at any j transfers control to statement 420.

The third statement following 417, namely the statement "IF (KPF.GT.10) GO to 418" transfers control to the U_j-test merely provided $\alpha_F = 1.0$, even if $j < 2$. But, the rationale of the U_j distribution leaves U_j undefined for $j < 2$. The possibility of a transfer of control to the U_j- distribution with $j = 1$ was provided for by setting the critical values of U_j equal to 2.0 for $j = 1$ and all values of $\alpha_U < 1.0$. Thus, if $j = 1$ then obviously $m_p = 0$ and the test statistic is (from table 1)

$$u_1 = \frac{2.0\, z_{(1)}}{SS_b + z_{(1)}} = \frac{2.0}{1 + SS_b/z_{(1)}}$$

for case $b(n_0 = 1, m_p = 0)$ or

$$u_1 = \frac{n_0 z_{(1)}}{SS_c + z_{(1)}}$$

for case c $(n_0 > 1, m_p = 0)$. Thus, for case b, $u_{(1)} \leq 2.0$ for $SS_b \geq 0.0$ and $z_{(1)}$ would not test as significant. For case c $(n_0 > 1, m_p = 0)$, $u_{(1)} > 2.0$ only if

$$\frac{n_0 z_{(1)}}{SS_c + z_{(1)}} > 2.0$$

hence only if

$$z_{(1)} > \frac{2.0\ SS_c}{(n_0 - 2.0)}$$

Let σ^2 be estimated by $SS_c/(n_0 - 1)$. Then $u_{(1)} > 2.0$ only if

$$z_{(1)} > \frac{2.0(n_0 - 1)}{(n_0 - 2.0)} \cdot \frac{SS_c}{(n_0 - 1)} > C(n_0)\hat{\sigma}^2$$

The following table shows $C(n_0)$ as a function of n_0 for the values of n_0 appropriate to case c.

n_0	$C(n_0)$
2	∞
3	4.0
4	3.0
5	2.67
6	2.5

Thus, $z_{(1)}$(which is the smallest of the ordered mean squares) would have to be much larger than $\hat{\sigma}^2$ before $z_{(1)}$ would be declared significant.

The flow chart for the model deletion and for the estimate, $\hat{\rho}$, is shown by figure 2(c). Transfer of control to statement 420, 421, or 422 leads to the estimate, respectively:

$$\hat{\eta} = \text{integer} \leq r_\eta(j - 1)$$

or

$$\hat{\eta} = \text{integer} \leq r_\eta m_p$$

or

$$\hat{\eta} = \text{integer} \leq r_\eta(n_c - 1)$$

With $\hat{\eta}$ so estimated, the $\hat{\eta}$ smallest absolute value coefficients (beyond b_1) are set equal to zero with the statements ending at 425.

<u>Section 5. - Predictions</u>. - Predicted values of the dependent variable for all the treatments of the fractional factorial experiment are computed in this section using SUBROUTINE YATES' and the reversed Yates' method as proposed by Duckworth (1965).

The operation of Yates' method followed by the "reversed Yates' method" is illustrated by the following table for a 2^2 experiment:

YATES METHOD

YOBS		B	B/FNC
Y_1	$Y_1 + Y_2$	$Y_1 + Y_2 + Y_3 + Y_4$	$(Y_1 + Y_2 + Y_3 + Y_4)/4$
Y_2	$Y_3 + Y_4$	$Y_2 - Y_1 + Y_4 - Y_3$	$(Y_2 - Y_1 + Y_4 - Y_3)/4$
Y_3	$Y_2 - Y_1$	$Y_3 + Y_4 - Y_1 - Y_2$	$(Y_3 + Y_4 - Y_1 - Y_2)/4$
Y_4	$Y_4 - Y_3$	$Y_4 - Y_3 - Y_2 + Y_1$	$(Y_4 - Y_3 - Y_2 + Y_1)/4$

REVERSED YATES METHOD

YOBS		B	YPRED
$(Y_4 - Y_3 - Y_2 + Y_1)/4$	$(Y_4 - Y_2)/2$	Y_4	Y_1
$(Y_3 + Y_4 - Y_1 - Y_2)/4$	$(Y_2 + Y_4)/2$	Y_3	Y_2
$(Y_2 - Y_1 + Y_4 - Y_3)/4$	$(Y_3 - Y_1)/2$	Y_2	Y_3
$(Y_1 + Y_2 + Y_3 + Y_4)/4$	$(Y_1 + Y_3)/2$	Y_1	Y_4

In the case of the computer program, there are n_c parameters estimated from a fractional factorial experiment.

<u>Section 6. - Accumulation of errors</u>. - The squared error for each prediction is accumulated (as required by eq. (24)) in the array ERSQ(I,L,M) as computed with the loop "DO 609 I = 1, NC." These accumulations are stored for each combination of L and M as indicated by the loops terminating at statements 680 and 690, and this process is repeated for each of the n_e sets of random numbers as indicated by the loop terminating at statement 699. For the purpose of computing the variance of the squared error of prediction, the quantity

$$\sum_{n=1}^{n_e} (e^2_{pi\ell mn})^2$$

of equation (27) is computed within the loop ending at statement 609 and stored as ERSQSQ(I,L,M).

<u>Section 7. - Determination of maximum and mean squared errors and their variances</u>. - The purpose of this section is to determine maximums and means of the prediction errors over the space of the experiment after the errors have been evaluated over that space by accumulating over the simulations. The accumulation over the number, n_e, of simulations had been stored in the array ERSQ(I,L,M). For particular L, and M, the determination of the largest prediction error over the space of the experiment as defined by equation (25) is done through repeated use of the library subroutine AMAX1, which determines a real number as a function of two real arguments. This is done within the loop

"DO 750 I = 1, NC." The summation for the mean squared prediction error over the space of the experiment as required by equation (26) is also done within the same loop terminating at statement 750. After division by the appropriate divisors, these two evaluations of error are stored in the arrays (ERSQMX(L,M) and AVERSQ(L,M). The quantity

$$\sum_{n=1}^{n_e} (e^2_{pi\ell mn})^2 - \frac{1}{n_e}\left(\sum_{n=1}^{n_e} e^2_{pi\ell mn}\right)^2$$

is computed and stored as TEM within the loop ending at statement number 750. The quantity $\hat{V}(e^2_{\ell m})_{max}$ defined by equation (28) is determined to be the maximum of the values of TEM as determined by

E = AMAX1(E,TEM)

and from this maximum, $\hat{V}(e^2_{\ell m})_{max}$ is computed and stored with the statement

VESQMX(L,M) = E/FNEM1

The sum of the values of TEM as given by

F = F + TEM

is then used to compute $\hat{V}(\overline{e}^2_{\ell m})$ according to equation (29) using the statement

AVVESQ(L,M) = F/FEM1NC

The computation ends if the data for MP, KPF, KPU, RF, and RETA, are exhausted; otherwise a new strategy is investigated by returning control to statement 50.

Section 8. - Output. - The output is illustrated in appendix E. The NAMELIST output was incorporated only for program checking.

Section 9. - Yates' method subroutine. - This subroutine is essentially that of part of the main program of POOLMS (Amling and Holms (1973)) except with the last few statements modified so that the subroutine can be used for the direct Yates' method and also for the reversed Yates' method; as was also done in POOLES (Holms (1974)).

The algorithm for Yates' method is described as follows: The "observations" $y_{i,j}$ may be visualized as a column ($j = 1$) with row index $i = 1,\ldots,2^\ell$. The column is then operated on according to Yates' method to produce a succession of columns $j = 2,\ldots,\ell$. The successive columns for any k^{th} row are computed as follows:

$$y_{k,j} = y_{i+1,j-1} + y_{i,j-1} \begin{cases} i = 1,\ 3,\ 5,\ldots,2^\ell - 1 \\ k = (i + 1)/2 \end{cases}$$

$$y_{k,j} = y_{i+1,j-1} - y_{i,j-1} \begin{cases} i = 1, 3, 5, \ldots, 2^{\ell} - 1 \\ k = (2^{\ell} + i + 1)/2 \end{cases}$$

New columns are computed according to the two preceding equations for $j = 2, \ldots, \ell$ (to create ℓ columns).

APPENDIX D

LISTING OF COMPUTER PROGRAM POOL9U

```
C
C      1A.- DECLARATIONS AND TABLES
C
       DIMENSION REMARK(14), ALPHA(11), TB(16,10), RN(24), IND(16),Z(16),
      1THETA(5), ND(7), DELTA(15), YMJ(16,5),  AVRHO(5,7), ERSQ(16,5,7),
      2ERSQSQ(16,5,7),ERSQMX(5,7), COERMX(5,7), AVERSQ(5,7),
      3VESQMX(5,7), AVVESQ(5,7), ADCOER(5,7), BFM(16),FTB(20,10)
C
       COMMON KK, YOBS(16), B(16)
C
       DATA (ALPHA(I),I=1,11)/0.001,0.002,0.005,0.01,0.025,0.05,0.10,
      10.25,0.50,0.75,1.0/
C
       DATA((FTB(I,J),J=1,10),I=1,20)/405280.0,101321.3,16211.0,4052.2,64
      A7.8,161.4,39.86,5.626,1.000,0.1716,998.5,498.5,198.5,98.50,38.51,1
      B8.51,8.526,2.571,0.6667,0.1333,167.0,104.3,55.55,34.12,17.44,10.13
      C,5.538,2.024,0.5851,0.1220,74.14,51.45,31.33,21.20,12.22,7.709,4.5
      D45,1.807,0.5486,0.1165,47.18,34.73,22.78,16.26,10.01,6.608,4.060,1
      E.692,0.5281,0.1134,35.51,27.12,18.64,13.74,8.813,5.987,3.776,1.621
      F,0.5149,0.1113,29.24,22.90,16.24,12.25,8.073,5.591,3.589,1.573,0.5
      G057,0.1099,25.42,20.26,14.69,11.26,7.571,5.318,3.458,1.538,0.4990,
      H0.1088,22.66,18.46,13.61,10.56,7.209,5.117,3.360,1.512,0.4938,0.10
      I80,21.04,17.17,12.63,10.04,6.937,4.965,3.285,1.492,0.4897,0.1073,1
      J9.69,15.20,12.23,9.646,6.724,4.844,3.225,1.475,0.4864,0.1068,18.64
      K,15.44,11.75,9.330,6.554,4.747,3.175,1.461,0.4837,0.1063,17.82,14.
      L84,11.37,9.074,6.414,4.667,3.136,1.450,0.4814,0.1059,17.14,14.34,1
      M1.06,8.862,6.298,4.600,3.102,1.4403,0.4794,0.1056,16.59,13.94,10.8
      N0,8.683,6.200,4.543,3.073,1.432,0.4778,0.1053,16.12,13.59,10.58,8.
      O531,6.115,4.494,3.048,1.425,0.4763,0.1051,15.72,13.29,10.38,8.400,
      P8.042,4.4513,3.026,1.419,0.4750,0.1049,15.38,13.03,10.22,8.285,5.9
      Q78,4.414,3.007,1.4130,0.4736,0.1047,15.06,12.81,10.07,8.185,5.922,
      R4.361,2.990,1.408,0.4728,0.1045,14.82,12.62,9.944,8.096,5.872,4.35
      S1,2.975,1.404,0.4719,0.1044/
C
       DATA ((TB(I,J),J=1,10),I=1,16)/2.0,2.0,2.0,2.0,2.0,2.0,2.0,2.0,2.0
      1,2.0,2.0000,1.99999,1.99997,1.99986,1.99917,1.99687,1.9877,1.923,1
      2.706,1.382,2.9976,2.9960,2.9904,2.9809,2.951,2.904,2.805,2.527,2.0
      386,1.688,3.976,3.962,3.925,3.870,3.750,3.625,3.412,2.949,2.395,1.9
      461,4.887,4.845,4.755,4.65,4.44,4.21,3.89,3.267,2.658,2.184,5.74,5.
      563,5.46,5.31,4.99,4.68,4.28,3.57,2.893,2.371,6.51,6.33,6.11,5.87,5
      6.45,5.09,4.61,3.83,3.11,2.54,7.20,6.96,6.65,6.35,5.88,5.44,4.91,4.
      706,3.29,2.69,7.81,7.52,7.10,6.78,6.25,5.75,5.17,4.27,3.45,2.82,8.3
      84,8.01,7.53,7.17,6.59,6.03,5.41,4.45,3.60,2.95,8.82,8.44,7.95,7.53
      9,6.89,6.28,5.61,4.62,3.74,3.07,9.26,8.84,8.33,7.87,7.13,6.50,5.81,
      A4.77,3.87,3.17,9.67,9.21,8.68,8.16,7.37,6.71,5.99,4.92,3.99,3.27,1
      B0.05,9.55,8.95,8.42,7.59,6.91,6.15,5.05,4.10,3.37,10.40,9.86,9.20,
      C8.66,7.79,7.07,6.30,5.17,4.20,3.46,10.72,10.14,9.43,8.83,7.96,7.23
      D,6.44,5.29,4.30,3.55/
C
       NAMELIST /OUT1/ B /OUT2/ YOBS /OUT3/ YMJ /OUT4/ INDXN /OUT5/ RN
       NAMELIST /OUT6/ INDXM /OUT7/NT /OUT8/ INDXL /OUT9/ IND,Z
       NAMELIST /OUT10/ NDF,TEM /OUT11/ NU /OUT12/ SSJYO /OUT14/ ETA
       NAMELIST /OUT13/ INDXJ, TEM, NDF, TEST, JN /OUT15/JETA
       NAMELIST /OUT16/ RHO /OUT17/ ERSQ /OUT18/ ERSQSQ /OUT19/ RTENT
       NAMELIST /OUT20/ KODE /OUT21 / BFM
C
```

```
C     1B.- INPUTS AND CONSTANTS
C
      READ(5,800) (REMARK(I)-I=1,14)
      WRITE(6,801) (REMARK(I),I=1,14)
      READ (5,802) LGMH, NE, KODE
      IF (KODE .GT. 0 ) WRITE (6, OUT20)
      READ(5,804) LTH, (THETA(L), L=1,LTH)
      READ (5,806) NDELTA, (DELTA(K), K=1,NDELTA)
      WRITE (6,807) NDELTA, (DELTA(K), K=1,NDELTA)
      READ (5,810) NNO, (NO(M), M=1,NNO)
      KK = LGMH
      NC = 2**LGMH
      WRITE (6,803) KK, NC, NNO, NE
      NCM1 = NC-1
      NCM2 = NC-2
      NCP1 = NC+1
      NTMX = NC + NO(NNO)
      NTMXP1 = NTMX + 1
      NEM1 = NE - 1
      FNC =  NC
      FNE = NE
      FNEM1 = NEM1
      FNEBNC = NE*NC
      FEM1NC = NEM1*NC
C
C     1C.- POPULATION MEANS
C
      DO 10 I=1,NC
      B(I) = 0.0
   10 CONTINUE
      DO 20 I=1,NDELTA
      B(I+1) = DELTA(I)
   20 CONTINUE
      IF (KODE .GT. 0 ) WRITE (6, OUT1 )
      DO 22 I=1,NC
      NCP1MI = NCP1-I
      YOBS(I) = B(NCP1MI)
   22 CONTINUE
      CALL YATES
      IF (KODE .GT. 0 ) WRITE (6, OUT1 )
      DO 30 I=1,NC
      NCP1MI = NCP1-I
      YOBS(I) = B(NCP1MI)
   30 CONTINUE
      IF (KODE .GT. 1 ) WRITE (6, OUT2 )
      DO 48  L=1,LTH
      DO 47 I=1,NC
      YMU(I,L) = YOBS(I)*THETA(L)
   47 CONTINUE
   48 CONTINUE
      IF (KODE .GT. 2 ) WRITE (6, OUT3 )
C
C     1D. - STRATEGY
C
   50 READ (5,806,END = 899) MP,KPF,KPJ,RF,RETA
      MPP1 = MP+1
      CALL SAND (XS)
```

```
      DO 99 M=1,NNO
      DO 98 L=1,LTH
      AVRHO(L,M)= 0.0
      DO 97 I=1,NC
      ERSC(I,L,M)= 0.0
      ERSQSQ(I,L,M) = 0.0
   97 CONTINUE
   98 CONTINUE
   99 CONTINUE
C
C     2.- SIMULATIONS AND MODEL FITTING
C
      DO 699  N=1,NE
      INDXN = N
      IF (KODE .GT. 3 ) WRITE (6, OUT4 )
      DO 213 I=1,NTMXP1
      CALL RAND(RN(I))
  213 CONTINUE
      IF (KODE .GT. 4 ) WRITE (6, OUT5 )
      DO 215 I=1,NTMX,2
      E= SQRT(-2.0*ALOG(RN(I)))
      D= 6.2831853*RN(I+1)
      RN(I)= E*COS(D)
      RN(I+1)= E*SIN(D)
  215 CONTINUE
      IF (KODE .GT. 4 ) WRITE (6, OUT5 )
      DO 690 L=1,LTH
      INDXL = L
      IF (KODE .GT. 7 ) WRITE (6, OUT8 )
      DO 224 I=1,NC
      YOBS(I) = YMU(I,L) + RN(I)
  224 CONTINUE
      IF (KODE .GT. 1 ) WRITE (6, OUT2 )
      CALL YATES
      IF (KODE .GT. 0 ) WRITE (6, OUT1 )
C
C     3.- CONSTRUCTION AND ORDERING OF MEAN SQUARES
C
      DO 309 I=1,NC
      IND(I)= I
      Z(I) = B(I+1)*B(I+1)/FNC
      BFM(I) = B(I)/FNC
  309 CONTINUE
      IF (KODE .GT. 8 ) WRITE (6, OUT9 )
      IF (KODE .GT. 0 ) WRITE (6, OUT21 )
C
      DO 313 J=1,NCM2
      TEST = Z(NCM1)
      IN = NCM1
      DO 312 NA=J,NCM2
      IF(TEST-Z(NA))  312,312,311
  311 TEST = Z(NA)
      IN= NA
  312 CONTINUE
      ITEM= IND(IN)
      TEM= Z(IN)
      IND(IN)= IND(J)
```

```
      Z(IN)= Z(J)
      IND(J)= ITEM
      Z(J) = TEM
  313 CONTINUE
      IF (MODE .GT. 8 ) WRITE (6, OUT9 )
      DO 650 M=1,NNM
      INDXM = M
      IF (MODE .GT. 5 ) WRITE (6, OUT6 )
      IF (MODE .GT. 10) WRITE (6, OUT11)
      DO 316 I = 1,NC
      B(I) = BEM(I)
  316 CONTINUE
      NT = NC + NO(M)
      IF (MODE .GT. 6 ) WRITE (6, OUT7 )
      FNO = NO(M)
      RFNO = RF*FNO
      TY = 0.0
      IF (NO(M)-1) 330, 325, 320
  320 SSQYO = 0.0
      DO 322 I=NCP1,NT
      TY = TY + RN(I)
      SSQYO = SSQYO + RN(I)**2
  322 CONTINUE
      IF (MODE .GT. 11) WRITE (6, OUT12)
      TEM = SSQYO - ((TY**2)/FNO)
      NDF = NO(M)-1
      B(1) = (FNC*B(1) + TY)/(FNC + FNO)
      GO TO 355
  325 TY = RN(NCP1)
      TEM = .941176*((TY - BEM(1))**2)
      NDF = 1
      B(1) = (FNC*B(1) + TY)/(FNC + FNO)
      GO TO 355
  330 TEM = 0.0
      NDF = 0
  355 IF (MODE .GT. 5) WRITE (6, OUT6)
      IF ( MODE .GT. 7 ) WRITE (  6, OUT8 )
      IF (MODE .GT. 9 ) WRITE (6, OUT9 )
      IF (NO(M) .LT. 1 .AND. MP .LT. 1) GO TO 472
      IF (MPF .GT. 30 .AND. MPU .GT. 90) GO TO 421
      IF (MP .LT. 1) GO TO 417
      DO 365 J = 1,MP
      INDXJ = J
      TEM = TEM + Z(J)
  365 CONTINUE
      NDF = NDF + MP
      IF (MODE .GT. 9 ) WRITE (6, OUT10)
C
C     4.- DELETION OF TERMS
C
  417 DO 419 J=MPP1,NCM1
      INDXJ = J
      FJ = J
      IF (MPF .GT. 30) GO TO 418
      IF (FJ .GT. RFNO .AND. INDXJ .GT. 1) GO TO 418
C
C           F - TEST
```

```
C
      FNDF = NDF
      TEST = FNDF * Z(J) / TEM
      IF (KODE .GT. 12) WRITE (6, OUT13)
      IF (TEST .GT. FTB(NDF,KPF)) GO TO 420
      TEM = TEM + Z(J)
      NDF = NDF + 1
      GO TO 419
C
C           UJ - TEST
C
  418 IF (KPU .GT. 10) GO TO 420
      NDF = NDF + 1
      JN = J
      FNDF = NDF
      TEST = FNDF*Z(J)/(TEM+Z(J))
      IF (KODE .GT. 12) WRITE (6, OUT13)
      IF (TEST .GT. TB(JN,KPU)) GO TO 420
      TEM = TEM + Z(J)
  419 CONTINUE
      JETA = NCM1
      GO TO 422
  420 JETA = J-1
      IF (KODE .GT. 12) WRITE (6, OUT13)
      GO TO 422
  421 JETA = MP
  422 ETA = JETA
      IF (KODE .GT. 13) WRITE (6, OUT14)
      JETA = IFIX(RETA*ETA)
      IF (KODE .GT. 14) WRITE (6, OUT15)
      IF (JETA .LT. 1) GO TO 434
      DO 425 J=1,JETA
      INDXJ = J
      INDX = IND(J)+1
      B(INDX)= 0.0
  425 CONTINUE
      IF (KODE .GT. 0 ) WRITE (6, OUT1 )
      GO TO 434
  432 JETA = 0
  434 RHO = NCM1 - JETA
      IF (KODE .GT. 15) WRITE (6, OUT16)
      AVRHO(L,M) = AVRHO(L,M) + RHO
C
C     5.- PREDICTIONS
C
      IF (KODE .GT. 5 ) WRITE ( 6, OUT6 )
      IF ( KODE .GT. 7 ) WRITE (  6, OUT8 )
      DO 546 I=1,NC
      NCP1MI = NCP1-I
      YOBS(I) = B(NCP1MI)
  546 CONTINUE
      IF (KODE .GT. 1 ) WRITE (6, OUT7 )
      CALL YATES
      IF (KODE .GT. 0 ) WRITE (6, OUT1 )
C
C     6.- ACCUMULATION OF ERRORS
C
```

```
      DO 679 I=1,NC
      NCP1MI = NCP1-I
      TEM = (R(NCP1MI) - YMU(I,L))**2
      ERSQ(I,L,M) = ERSQ(I,L,M) + TEM
      ERSQSQ(I,L,M)= ERSQSQ(I,L,M) + TEM**2
  679 CONTINUE
  680 CONTINUE
  690 CONTINUE
      IF (KODE .GT. 16) WRITE (6, OUT17)
      IF (KODE .GT. 17) WRITE (6, OUT18)
  699 CONTINUE
      IF (KODE .GT. 16) WRITE (6, OUT17)
      IF (KODE .GT. 17) WRITE (6, OUT18)
C
C     7.- DETERMINATION OF MAXIMUM AND MEAN SQUARED ERRORS
C
      DO 790 M=1,NNC
      INDXM = M
      IF (KODE .GT. 5 ) WRITE ( 6, OUT6 )
      FNT = NC + NO(M)
      RTFNT = SQRT(FNT)
      IF (KODE .GT. 18) WRITE (6, OUT19)
      DO 760 L=1,LTH
      INDXL = L
      IF (KODE .GT. 7 ) WRITE (6, OUT8 )
      C = 0.0
      D = 0.0
      E= 0.0
      F= 0.0
      DO 750 I=1,NC
      C = AMAX1(C,ERSQ(I,L,M))
      D = D + ERSQ(I,L,M)
      TEM= ERSQSQ(I,L,M)- ((ERSQ(I,L,M))**2)/FNE
      E = AMAX1(E,TEM)
      F = F + TEM
  750 CONTINUE
      ERSQMX(L,M) = C/FNE
      COERMX(L,M) = (SQRT(ERSQMX(L,M)))/THETA(L)
      AVERSQ(L,M) = D/FNEBNC
      VESQMX(L,M)= E/FNEM1
      AVVESQ(L,M) = F/FEM1NC
      AVRHO(L,M) = AVRHO(L,M)/FNE
      ADCOER(L,M) = COERMX(L,M)*RTFNT
  760 CONTINUE
  790 CONTINUE
C
C     8.- OUTPUT
C
      WRITE (6,809) (MP, RF, ALPHA(KPF), ALPHA(KPU), BETA)
      WRITE (6,811) (NO(M), M=1,NNO)
      WRITE (6,813)
      WRITE (6,815)
      WRITE (6,817) (THETA(L),(AVRHO (L,M),M=1,NNC ),L=1,LTH)
      WRITE (6,831)
      WRITE (6,817) (THETA(L),(ERSQMX(L,M),M=1,NNO ),L=1,LTH)
      WRITE (6,835)
      WRITE (6,817) (THETA(L),(AVERSQ(L,M),M=1,NNC ),L=1,LTH)
```

```
      WRITE (6,836)
      WRITE (6,817) (THETA(L),(VESQMX(L,M),M=1,NNO ),L=1,LTH)
      WRITE (6,837)
      WRITE (6,817) (THETA(L),(AVVESQ(L,M),M=1,NNO ),L=1,LTH)
      WRITE (6,838)
      WRITE (6,817) (THETA(L),(COERMX(L,M),M=1,NNO ),L=1,LTH)
      WRITE (6,839)
      WRITE (6,817) (THETA(L), (ADCOER(L,M), M=1,NNO),L=1,LTH)
      GO TO 50
  899 STOP
C
  800 FORMAT (13A6,A2)
  801 FORMAT (1H1,//10X,13A6,A2//)
  802 FORMAT (3I5)
  803 FORMAT (1H0,3X,6HLGMH =I5,5X,4HNC =I5,5X,5HNNO =I5,5X,4HNE =I5)
  804 FORMAT (I8,5F8.3)
  806 FORMAT (I4/(10F8.5))
  807 FORMAT (1H0,5HRHO =I5,5X,7HDELTA =//(1X,10F10.5))
  808 FORMAT (3I5,2F5.3)
  809 FORMAT (1H1//,1X,4HMP =I5,5X,4HRF =F5.3,5X,8HALPHAF =F6.3,5X,8HALP
     AHAU =F6.3,5X,6HRETA =F6.3)
  810 FORMAT (8I2)
  811 FORMAT (1H0,4HNO =7I14//)
  813 FORMAT (1H0,5HTHETA)
  815 FORMAT  (1H0,20X,5HAVRHO//)
  817 FORMAT (1X,F8.3,7E14.4)
  831 FORMAT (1H0,20X,6HERSQMX//)
  835 FORMAT (1H0,20X,6HAVERSQ//)
  836 FORMAT (1H0,20X,6HVESQMX//)
  837 FORMAT (1H0,20X,6HAVVESQ//)
  838 FORMAT (1H0,20X,6HCOERMX//)
  839 FORMAT (1H0,20X,6HADCOER//)
      END
```

```
      SUBROUTINE YATES
C
C     9.- YATES METHOD SUBROUTINE
C
      COMMON KK,Y(16),B(16)
      II = 2**KK
      IIDB2 = II/2
      KKM1 = KK-1
      DO 908 K=1,KKM1
      DO 906 I=1,II,2
      IP1D2 = (I+1)/2
      B(IP1D2) = Y(I+1)+Y(I)
      LL = IP1D2+IIDB2
  906 B(LL) = Y(I+1)-Y(I)
      DO 907 I=1,II
  907 Y(I) = B(I)
  908 CONTINUE
      DO 909 I=1,II,2
      IP1D2 = (I+1)/2
      B(IP1D2) = Y(I+1)+Y(I)
      LL = IP1D2+IIDB2
      B(LL) = Y(I+1)-Y(I)
  909 CONTINUE
      RETURN
      END
```

APPENDIX E

ILLUSTRATIVE OUTPUT OF COMPUTER PROGRAM POOL9U

MAY 30, 1980 TEM = 0.941176*(WAS TEM = 0.5*(

RHO = 15 DELTA =

2.10819 1.66452 1.40939 1.22186 1.36945 .73855 .82213 .71606 .61764 .52503
.43685 .35211 .26985 .18921 .10835

LGMM = 4 NC = 16 NNO = 7 NE = 1000

MP = 0 RF = 3.000 ALPHAF = .500 ALPHAJ = .100 RETA = .800

NO =	0	1	2	3	4	5	6
THETA							

AVRHO

THETA	0	1	2	3	4	5	6
.125	.1500+02	.8442+01	.1224+02	.1224+02	.1194+02	.1166+02	.1149+02
.250	.1500+02	.9730+01	.1302+02	.1263+02	.1255+02	.1235+02	.1224+02
.500	.1500+02	.1154+02	.1389+02	.1361+02	.1342+02	.1333+02	.1325+02
1.000	.1500+02	.1410+02	.1456+02	.1447+02	.1443+02	.1437+02	.1434+02
2.000	.1500+02	.1490+02	.1491+02	.1489+02	.1489+02	.1489+02	.1489+02

ERSQMX

THETA	0	1	2	3	4	5	6
.125	.1066+01	.9285+00	.1041+01	.1038+01	.1035+01	.1026+01	.1023+01
.250	.1066+01	.1361+01	.1058+01	.1051+01	.1054+01	.1053+01	.1047+01
.500	.1066+01	.2850+01	.1068+01	.1075+01	.1073+01	.1067+01	.1070+01
1.000	.1066+01	.2947+01	.1068+01	.1059+01	.1061+01	.1056+01	.1053+01
2.000	.1066+01	.1119+01	.1065+01	.1062+01	.1061+01	.1055+01	.1049+01

AVERSQ

THETA	0	1	2	3	4	5	6
.125	.1006+01	.8985+00	.9820+00	.9935+00	.9937+00	.9882+00	.9850+00
.250	.1006+01	.1009+01	.1003+01	.9985+00	.9955+00	.9943+00	.9914+00
.500	.1006+01	.1239+01	.1009+01	.1004+01	.1002+01	.1001+01	.9962+00
1.000	.1006+01	.1241+01	.1004+01	.1003+01	.1001+01	.1000+01	.9963+00
2.000	.1006+01	.1022+01	.1003+01	.1001+01	.9979+00	.9960+00	.9934+00

VESQMX

THETA	0	1	2	3	4	5	6
.125	.2460+01	.1878+01	.2478+01	.2564+01	.2552+01	.2495+01	.2521+01
.250	.2460+01	.2735+01	.2545+01	.2571+01	.2573+01	.2520+01	.2577+01
.500	.2460+01	.1261+02	.2530+01	.2496+01	.2465+01	.2446+01	.2442+01
1.000	.2460+01	.6956+02	.2523+01	.2538+01	.2522+01	.2464+01	.2472+01
2.000	.2460+01	.2246+02	.2518+01	.2531+01	.2519+01	.2466+01	.2479+01

AVVESQ

THETA	0	1	2	3	4	5	6
.125	.2045+01	.1642+01	.1969+01	.2014+01	.1997+01	.1984+01	.1977+01
.250	.2045+01	.1950+01	.2032+01	.2017+01	.2007+01	.2002+01	.1997+01
.500	.2045+01	.3204+01	.2082+01	.2048+01	.2043+01	.2038+01	.2030+01
1.000	.2045+01	.7462+01	.2042+01	.2037+01	.2031+01	.2025+01	.2019+01
2.000	.2045+01	.3630+01	.2040+01	.2032+01	.2024+01	.2015+01	.2007+01

COERMX

THETA	0	1	2	3	4	5	6
.125	.8260+01	.7709+01	.8161+01	.8151+01	.8140+01	.8103+01	.8093+01
.250	.4130+01	.4667+01	.4114+01	.4101+01	.4107+01	.4099+01	.4092+01
.500	.2065+01	.3376+01	.2067+01	.2074+01	.2071+01	.2066+01	.2069+01
1.000	.1033+01	.1717+01	.1033+01	.1029+01	.1030+01	.1028+01	.1026+01
2.000	.5163+00	.5288+00	.5160+00	.5153+00	.5151+00	.5136+00	.5121+00

ADCOER

THETA	0	1	2	3	4	5	6
.125	.3304+02	.3178+02	.3462+02	.3553+02	.3640+02	.3713+02	.3796+02
.250	.1652+02	.1924+02	.1746+02	.1787+02	.1836+02	.1878+02	.1920+02
.500	.8260+01	.1392+02	.8770+01	.9039+01	.9264+01	.9469+01	.9703+01
1.000	.4130+01	.7079+01	.4384+01	.4484+01	.4605+01	.4709+01	.4814+01
2.000	.2065+01	.2180+01	.2189+01	.2246+01	.2304+01	.2354+01	.2402+01

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Table 1. - Expressions for degrees of freedom, sums of squares and test statistics.

Case	n_0	m_p	n_{df0}		Eq. No.	F-Test (See Eq. (14))	U_j-Test (See Eq. (16))
a	0	0				$(1 \le j \le r_f n_0)$	$1 \leqq j \leqq n_c - 1)$
b	1	0	1	$SS_b = 0.941176(y_{01} - b_1)^2$	(B4)	$f_{1,j} = \frac{(1 + j - 1)z_{(j)}}{SS_b + z_{(1)} + \cdots + z_{(j-1)}}$	$u_j = \frac{(n_{df0} + j)z_{(j)}}{SS_b + z_{(1)} + \quad + z_{(j)}}$
c	>1	0	$n_0 - 1$	$SS_c = \sum_{k=1}^{n_0} y_{0k}^2 - \frac{1}{n_0}\left(\sum_{k=1}^{n_0} y_{0k}\right)^2$	(B5)	$f_{1,n_0+j-2} = \frac{(n_0 - 1 + j - 1)z_{(j)}}{SS_c + z_{(1)} + \cdots + z_{(j-1)}}$	$u_j = \frac{(n_{df0} + j)z_{(j)}}{SS_c + z_{(1)} + \cdots + z_{(j)}}$
d	0	>0	m_p	$SS_d = \sum_{j=1}^{m_p} z_{(j)}$	(B7)		$u_j = \frac{j\, Z_{(j)}}{SS_d + z_{(mp+1)} + \cdots + z_{(j)}}$
e	1	>0	$1 + m_p$	$SS_e = SS_b + SS_d$	(B9)	$f_{1,j} = \frac{(1 + j - 1)z_{(j)}}{SS_e + z_{(mp+1)} + \cdots + z_{(j-1)}}$	$u_j = \frac{(1 + j)z_{(j)}}{SS_e + z_{(mp+1)} + \cdots + z_{(j)}}$
f	>1	>0	$n_0 - 1 + m_p$	$SS_f = SS_c + SS_d$	(B11)	$f_{1,n_0+j-2} = \frac{(n_0 - 1 + j - 1)z_{(j)}}{SS_f + z_{(mp+1)} + \cdots + z_{(j-1)}}$	$u_j = \frac{(n_0 - 1 + j)z_{(j)}}{SS_f + z_{(mp+1)} + \cdots + z_{(j)}}$

Table 2. - Values of j at which transfer from F to U_j distribution occurs.

n_0 / r_F	0	1	2	3	4	5	6
0.0	2	2	2	2	2	2	2
0.4	2	2	2	2	2	3	3
0.5	2	2	2	2	3	3	4
0.7	2	2	2	3	3	4	5
0.9	2	2	2	3	4	5	6
1.0	2	2	3	4	5	6	7
2.0	2	3	5	7	9	11	13
4.0	2	5	9	13	n.t.	n.t.	n.t.
8.0	2	9	n.t.	n.t.	n.t.	n.t.	n.t.
16.0	2	n.t.	n.t.	n.t.	n.t.	n.t.	n.t.

[n.t.] No transfer occurs.

Table 3. - Parameter combinations, $\beta_{k+1} = \theta\delta_k$, $\beta_1 = 0$.

k	δ_k	θ				
		0.125	0.250	0.500	1.000	2.000
1	2.1082	0.2635	0.5270	1.0541	2.1082	4.2164
2	1.6645	.2081	.4161	.8322	1.6645	3.3290
3	1.4094	.1762	.3524	.7047	1.4094	2.8188
4	1.2219	.1527	.3055	.6110	1.2219	2.4438
5	1.0694	.1337	.2674	.5347	1.0694	2.1388
6	0.9386	0.1173	0.2346	0.4693	0.9386	1.8772
7	.8221	.1028	.2055	.4110	.8221	1.6442
8	.7161	.0895	.1790	.3580	.7161	1.4322
9	.6176	.0772	.1544	.3088	.6176	1.2352
10	.5250	.0656	.1312	.2625	.5250	1.0500
11	0.4368	0.0546	0.1092	0.2184	0.4368	0.8736
12	.3521	.0440	.0880	.1760	.3521	.7042
13	.2699	.0337	.0675	.1350	.2699	.5398
14	.1892	.0237	.0473	.0946	.1892	.3784
15	.1084	.0136	.0271	.0542	.1084	.2168
$\Sigma\beta_k$:		1.556	3.112	6.225	12.449	24.898
$(\Sigma\beta_k)^{-1}$:		0.643	0.321	0.161	0.080	0.040

Table 4. - Admissible strategies and their operating characteristics.

(a) $n_0 = 0$.

m_p	r_F	α_F	α_U	r_η \ θ	$\bar{\rho}$		$V(e^2)_{max}$		$C_{ae,mx}$		R	
					0.125	2.0	0.125	2.0	0.125	2.0	0.125	2.0
0	--	1.0	1.0	0.0	15.00	15.00	2.460	2.460	33.04	2.065	1.1215	1.0000
1	--	1.0	.75	.10	14.98	14.98	2.462	2.605	33.02	2.072	1.1208	1.0034
1	--	1.0	.75	.20	14.92	14.91	2.453	2.504	33.00	2.084	1.1202	1.0092
1	--	1.0	.75	.40	14.72	14.66	2.450	2.585	32.99	2.133	1.1198	1.0329
1	--	1.0	.50	.15	14.72	14.55	2.456	2.791	32.97	2.170	1.1191	1.0508
1	--	1.0	.50	.20	14.64	14.49	2.443	2.812	32.96	2.181	1.1188	1.0562
1	--	1.0	.50	.25	14.39	14.17	2.442	2.975	32.95	2.240	1.1185*	1.0847*
1	--	1.0	.75	.54	14.43	14.36	2.450	7.879	32.94	2.375	1.1181	1.1501
1	--	1.0	.75	.55	14.42	14.36	2.454	8.715	32.92	2.404	1.1174	1.1642
1	--	1.0	.50	.35	14.04	13.75	2.443	8.117	32.90	2.808	1.1168	1.3598
1	--	1.0	.50	.40	13.96	13.60	2.422	13.75	32.88	3.178	1.1161	1.5390
1	--	1.0	.50	.45	13.78	13.34	2.398	34.87	32.70	3.910	1.1100	1.8935
1	--	1.0	.50	.50	13.36	12.89	2.458	98.42	32.58	4.931	1.1059	2.3879
1	--	1.0	.50	.55	13.26	12.79	2.429	167.5	32.55	5.479	1.1049	2.6533
1	--	1.0	.50	.75	12.57	11.88	2.383	2135.0	32.46	10.00	1.1018	4.8426
1	--	1.0	.50	.80	12.50	11.74	2.409	3159.0	32.40	11.11	1.1000	5.3801
1	--	1.0	.50	.85	12.32	11.57	2.299	4735.0	32.28	12.23	1.0957	5.9225
1	--	1.0	.50	.90	12.20	11.36	2.270	9239.0	32.27	14.23	1.0954	6.8910
1	--	1.0	.50	.95	12.05	11.13	2.253	18610.0	32.03	16.87	1.0872	8.1695
1	--	1.0	.05	.70	6.029	5.710	1.568	770.8	30.53	19.05	1.0363	9.2252
1	--	1.0	.10	.80	6.209	5.447	1.631	2632.0	30.39	22.36	1.0316	10.8281
1	--	1.0	.05	.75	5.137	4.783	1.472	1581.0	29.72	23.17	1.0088	11.2203
1	--	1.0	.025	.75	4.488	4.389	1.465	903.1	29.61	23.63	1.0051	11.4431
5	--	1.0	.05	.75	4.291	4.000	1.497	172.6	29.46	24.09	1.000	11.6659

*Security regret strategy.

Table 4. - Cont'd.

(b) $n_0 = 1$.

m_p	r_F	α_F	α_U	$r_\eta \diagdown \theta$	$\bar{p}$ 0.125	$\bar{p}$ 2.0	$V(e^2)_{max}$ 0.125	$V(e^2)_{max}$ 2.0	$C_{ae,mx}$ 0.125	$C_{ae,mx}$ 2.0	R 0.125	R 2.0
0	2.0	0.50	0.75	0.65	10.66	14.90	2.291	2.512	33.05	2.118	1.0875	1.0000
0	3.0	.50	.25	.80	8.747	14.91	1.898	2.508	32.04	2.119	1.0543	1.0005
0	3.0	.50	.10	.80	8.442	14.90	1.878	22.46	31.78	2.180	1.0457*	1.0293*
0	2.0	.50	.25	.75	7.463	14.65	1.714	408.0	31.61	4.094	1.0401	1.9330
0	2.0	.50	.10	.80	6.741	14.38	1.692	1029.0	31.20	5.989	1.0267	2.8277
0	0.0	.50	.10	.75	5.780	12.27	1.581	3796.0	30.82	12.17	1.0141	5.7460
0	0.5	.25	.05	.80	5.209	10.60	1.487	5046.0	30.60	15.57	1.0069	7.3513
0	4.0	.01	.05	.75	4.305	4.737	1.551	1607.0	30.53	23.90	1.0046	11.2842
0	4.0	.01	.025	.75	4.194	4.657	1.536	1426.0	30.51	24.02	1.0039	11.3409
0	8.0	.01	.05	.75	4.281	4.649	1.536	1408.0	30.47	24.03	1.0026	11.3456
0	4.0	.005	.05	.75	4.275	4.404	1.539	1024.0	30.46	24.30	1.0023	11.4731
0	4.0	.005	.025	.75	4.164	4.316	1.524	811.3	30.44	24.43	1.0016	11.5345
0	8.0	.005	.05	.775	4.209	4.276	1.517	716.3	30.39	24.50	1.0000	11.5675

*Security regret strategy.

Table 4. - Cont'd.

(c) $n_0 = 2$.

m_p	r_F	α_F	α_U	$r_\eta \backslash \theta$	$\bar{\rho}$		$V(e^2)_{max}$		$C_{ae,mx}$		R	
					0.125	2.0	0.125	2.0	0.125	2.0	0.125	2.0
0	2.0	0.75	0.50	0.90	13.12	14.98	2.437	2.519	34.40	2.187	1.1058	1.0000
0	1.0	.50	.75	.70	10.10	14.88	2.340	2.533	33.98	2.188	1.0923	1.0005
0	2.0	.50	.25	.85	9.957	14.91	2.048	2.518	33.20	2.189	1.0672	1.0009
0	3.0	.25	.50	.85	8.959	14.42	2.227	7.804	33.14	2.252	1.0653*	1.0297*
0	1.0	.50	.50	.74	8.510	14.83	2.180	31.24	33.08	2.466	1.0633	1.1276
0	1.0	.50	.50	.80	8.142	14.81	2.157	65.72	32.93	2.610	1.0585	1.1934
0	0.0	.75	.50	.80	8.045	14.61	2.157	142.3	32.91	3.152	1.0579	1.4412
0	2.0	.25	.25	.75	7.508	14.28	1.922	384.4	32.70	4.022	1.0511	1.8390
0	2.0	.25	.25	.80	7.327	14.28	1.895	416.8	32.57	4.096	1.0469	1.8729
0	1.0	.50	.10	.80	6.492	14.43	1.695	960.4	32.04	5.921	1.0299	2.7074
0	1.0	.50	.025	.75	6.361	14.30	1.674	1221.0	31.96	6.638	1.0273	3.0352
0	0.0	.50	.10	.80	5.362	12.30	1.580	3680.0	31.72	12.35	1.0196	5.6470
0	0.0	.50	.05	.80	5.206	12.04	1.548	3958.0	31.66	13.04	1.0177	5.9625
0	0.0	.50	.025	.75	5.136	11.91	1.541	4106.0	31.59	13.38	1.0154	6.1180
0	1.0	.25	.025	.80	5.176	11.24	1.533	4676.0	31.57	14.85	1.0148	6.7901
0	0.0	.25	.05	.80	5.045	10.46	1.524	5018.0	31.56	16.18	1.0145	7.3983
0	0.0	.25	.025	.75	4.976	10.30	1.517	5086.0	31.48	16.51	1.0119	7.5492
0	0.0	.10	.025	.75	4.909	9.569	1.516	5249.0	31.42	17.77	1.0100	8.1253
0	0.0	.10	.01	.75	4.840	9.521	1.496	5262.0	31.37	17.87	1.0084	8.1710
0	0.0	.10	.01	.80	4.829	9.521	1.499	5262.0	31.37	17.87	1.0084	8.1710
0	0.0	.025	.025	.75	4.864	9.292	1.517	5264.0	31.34	18.22	1.0074	8.3310
0	0.0	.025	.01	.75	4.795	9.244	1.497	5271.0	31.29	18.32	1.0058	8.3768
0	0.0	.025	.01	.80	4.784	9.244	1.500	5271.0	31.29	18.32	1.0058	8.3768
0	2.0	.002	.10	.80	4.362	4.436	1.491	1173.0	31.26	24.88	1.0048	11.3763
0	2.0	.005	.05	.75	4.276	4.349	1.480	915.8	31.24	25.06	1.0042	11.4586
0	2.0	.002	.05	.75	4.245	4.240	1.468	707.4	31.12	25.20	1.0003	11.5226
0	4.0	.002	.05	.80	4.086	4.113	1.478	394.2	31.11	25.41	1.0000	11.6187

*Security regret strategy.

Table 4. - Cont'd.

(d) $n_0 = 3$.

m_p	r_F	α_F	α_U	$r_\eta \backslash \theta$	$\bar{\rho}$		$V(e^2)_{max}$		$C_{ae,mx}$		K	
					0.125	2.0	0.125	2.0	0.125	2.0	0.125	2.0
0	1.0	0.75	0.50	0.85	10.53	14.97	2.119	2.523	34.41	2.244	1.0770	1.0000
0	1.0	.50	.50	.80	7.456	14.89	1.888	2.532	33.63	2.246	1.0526	1.0009
0	0.0	.75	.50	.80	6.581	14.73	1.691	11.95	33.29	2.301	1.0419*	1.0254*
0	0.7	.50	.50	.80	6.584	14.83	1.665	30.94	33.20	2.356	1.0391	1.0499
0	1.0	.50	.10	.80	6.418	14.86	1.706	64.86	32.94	2.578	1.0310	1.1488
0	1.0	.50	.05	.80	6.372	14.86	1.705	78.33	32.87	2.684	1.0289	1.1961
0	0.7	.50	.10	.75	5.195	14.44	1.520	833.1	32.39	5.725	1.0138	2.5512
0	0.7	.50	.025	.75	4.988	14.28	1.470	1161.0	32.21	6.689	1.0081	2.9808
0	0.0	.50	.025	.75	4.387	12.31	1.491	3665.0	32.19	12.63	1.0075	5.6283
0	0.7	.25	.05	.80	4.354	12.24	1.483	3765.0	32.13	12.79	1.0056	5.6996
0	0.7	.25	.025	.75	4.360	12.08	1.475	3957.0	32.09	13.26	1.0044	5.9091
0	0.0	.25	.025	.75	4.298	11.12	1.474	4662.0	32.06	15.20	1.0034	6.7736
0	1.0	.10	.025	.75	4.281	9.967	1.463	5162.0	32.02	17.16	1.0022	7.6471
0	2.0	.001	.05	.80	4.067	4.117	1.486	427.6	31.97	26.08	1.0006	11.6221
0	2.0	.001	.025	.75	4.075	4.097	1.477	371.7	31.95	26.12	1.0000	11.6399

*Security regret strategy.

Table 4. - Cont'd.

(e) $n_0 = 4$.

m_p	r_F	α_F	α_U	$r_\eta \backslash \theta$	$\bar{\rho}$		$V(e^2)_{max}$		$C_{ae,mx}$		R	
					0.125	1.0	0.125	2.0	0.125	2.0	0.125	2.0
0	0.5	0.75	0.50	0.85	7.038	14.96	1.845	2.501	34.38	2.300	1.0536	1.0000
0	1.0	.50	.05	.80	6.992	14.89	1.864	2.519	33.93	2.304	1.0398	1.0017
0	0.5	.50	.50	.80	5.904	14.83	1.605	2.533	33.89	2.309	1.0386*	1.0039*
0	0.0	.50	.50	.80	5.616	14.48	1.591	12.77	33.71	2.421	1.0331	1.0526
0	1.0	.25	.10	.80	4.830	14.48	1.639	151.2	33.12	3.096	1.0150	1.3461
0	0.0	.50	.25	.80	4.723	14.04	1.539	588.4	33.07	5.257	1.0135	2.2857
0	0.5	.50	.05	.75	4.737	14.44	1.486	854.5	32.94	5.930	1.0095	2.5783
0	0.9	.25	.05	.80	4.394	14.03	1.484	1131.0	32.90	6.710	1.0083	2.9174
0	0.0	.50	.05	.80	4.241	13.01	1.465	2700.0	32.74	10.69	1.0034	4.6478
0	0.5	.25	.05	.80	4.243	12.85	1.465	2923.0	32.67	11.15	1.0012	4.8478
0	0.0	.25	.05	.80	4.186	12.13	1.460	3653.0	32.64	12.86	1.0003	5.5913
0	0.9	.10	.05	.80	4.142	11.01	1.451	4547.0	32.63	15.11	1.0000	6.5696

*Security regret strategy.

Table 4. - Cont'd.

(f) $n_0 = 5$.

m_p	r_F	α_F	α_U	$r_\eta \diagdown \theta$	$\bar{\rho}$ 0.125	$\bar{\rho}$ 2.0	$V(e^2)_{max}$ 0.125	$V(e^2)_{max}$ 2.0	$C_{ae,mx}$ 0.125	$C_{ae,mx}$ 2.0	R 0.125	R 2.0
0	0.6	0.50	0.10	0.60	7.120	14.89	1.950	2.488	35.92	2.352	1.0719	1.0000
0	0.7	.50	.50	.80	6.049	14.89	1.739	2.486	34.81	2.354	1.0388	1.0009
0	0.7	.50	.10	.80	5.233	14.88	1.616	2.501	34.30	2.360	1.0236	1.0034*
0	1.0	.25	.10	.80	4.834	14.53	1.615	2.567	33.97	2.394	1.0137*	1.0179
0	0.8	.25	.25	.80	4.581	14.44	1.537	88.61	33.96	2.925	1.0134	1.2436
0	0.5	.50	.10	.75	4.616	14.68	1.508	253.0	33.90	3.826	1.0116	1.6267
0	0.5	.50	.10	.80	4.489	14.68	1.487	262.1	33.71	3.854	1.0060	1.6386
0	0.7	.25	.10	.80	4.247	14.30	1.492	468.3	33.70	4.703	1.0057	1.9996
0	0.5	.50	.05	.80	4.403	14.55	1.462	571.1	33.60	5.120	1.0027	2.1769
0	0.5	.50	.025	.80	4.388	14.51	1.462	691.4	33.58	5.540	1.0021	2.3554
0	0.5	.50	.01	.80	4.379	14.47	1.456	805.2	33.57	5.889	1.0018	2.5038
0	0.5	.25	.05	.80	4.101	13.27	1.466	2202.0	33.51	9.744	1.0000	4.1429

*Security regret strategy.

Table 4. - Concluded.

(g) $n_0 = 6$.

m_p	r_F	α_F	α_U	$r_\eta \diagdown \theta$	$\bar{\rho}$ 0.125	$\bar{\rho}$ 2.0	$V(e^2)_{max}$ 0.125	$V(e^2)_{max}$ 2.0	$C_{ae,mx}$ 0.125	$C_{ae,mx}$ 2.0	R 0.125	R 2.0
0	0.6	0.50	0.10	0.60	6.862	14.89	1.805	2.479	36.77	2.401	1.0736	1.0000
0	0.7	.50	.10	.80	6.071	14.89	1.786	2.478	35.38	2.402	1.0330	1.0004
0	0.5	.50	.05	.80	4.850	14.88	1.549	2.485	34.72	2.408	1.0137*	1.0029*
0	0.9	.25	.05	.80	4.548	14.55	1.679	2.536	34.66	2.437	1.0120	1.0150
0	0.7	.25	.10	.80	4.336	14.54	1.500	30.30	34.55	2.524	1.0088	1.0512
0	1.0	.10	.10	.80	4.239	13.75	1.499	97.63	34.48	3.274	1.0067	1.3636
0	0.5	.25	.10	.80	4.159	14.41	1.496	245.1	34.43	3.824	1.0053	1.5927
0	0.5	.25	.05	.80	4.080	14.33	1.493	461.5	34.34	4.754	1.0026	1.9800
0	0.4	.50	.025	.75	4.324	14.62	1.453	476.4	34.30	4.804	1.0015	2.0008
0	0.0	.50	.025	.80	4.037	13.65	1.477	1573.0	34.25	8.384	1.0000	3.4919

*Security regret strategy.

Table 5. - Values of $\bar{\rho}$ as functions of n_0, θ, and choice of strategy.

n_0	Strategy					
	[a]Minimum $C_{ae,mx}(2.0)$		[b]Security Regret		[c]Minimum $C_{ae,mx}(0.125)$	
	θ					
	0.125	2.0	0.125	2.0	0.125	2.0
0	15.00	15.00	14.39	14.17	4.291	4.000
1	10.66	14.90	8.442	14.90	4.209	4.276
2	13.12	14.98	8.959	14.42	4.086	4.113
3	10.53	14.97	6.581	14.73	4.075	4.097
4	7.038	14.96	5.904	14.83	4.142	11.01
5	7.120	14.89	4.834	14.53	4.101	13.27
6	6.862	14.89	4.850	14.88	4.037	13.65

[a]From first row of table 4.

[b]From asterisked results in table 4.

[c]From last row of table 4.

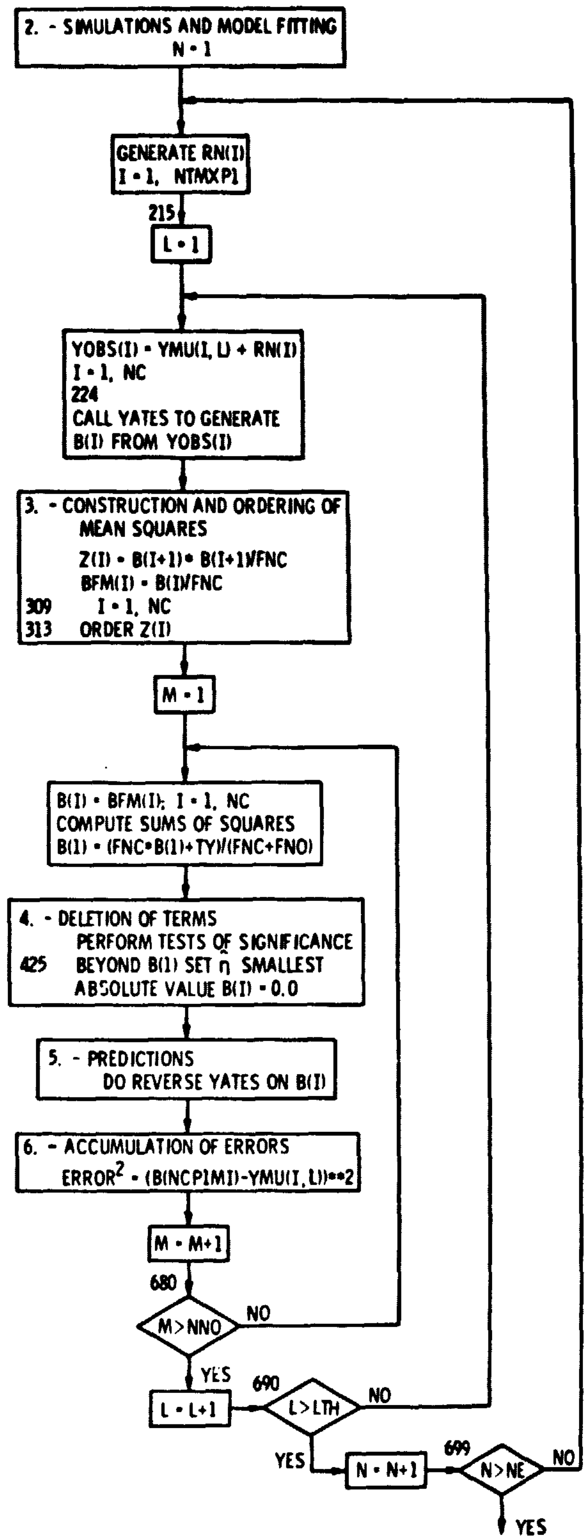
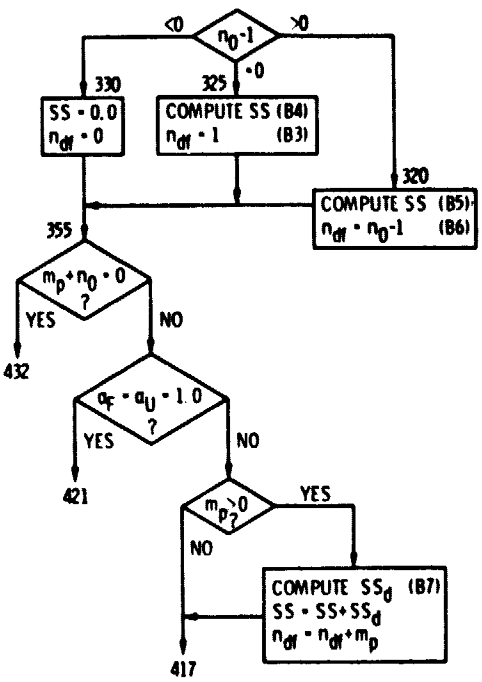
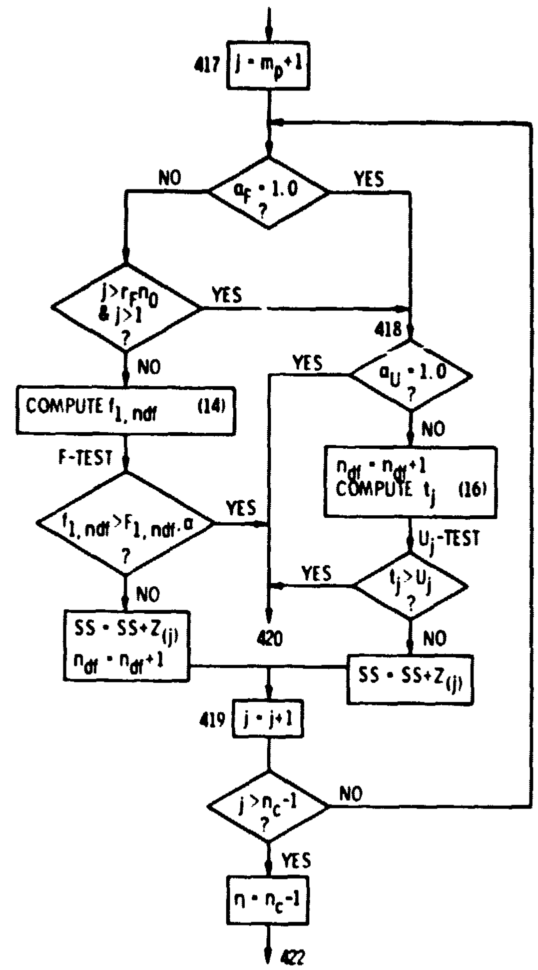


Figure 1. - Use of arrays YOBS(I), B(I), and BFM(I).



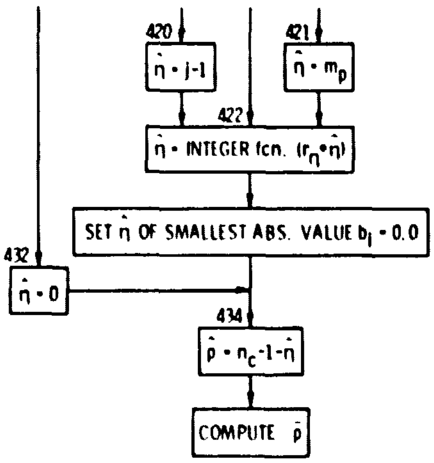
(a) Computation of sums of squares.

Figure 2. - Flow chart of program POOL9U. Numbers in ( ) are equation numbers of text. Three digit integers are statement numbers in Appendix D.



(b) Tests of significance.

Figure 2. - Continued.



(c) Deletion of insignificant coefficients.

Figure 2. - Concluded.